

FOURIER TRANSFORMS ON THE QUANTUM  $SU(1,1)$  GROUPERIK KOELINK AND JASPER V. STOKMAN  
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November 2, 1999

ABSTRACT. The main goal is to interpret the Askey-Wilson function and the corresponding transform pair on the quantum  $SU(1,1)$  group. A weight on the  $C^*$ -algebra of continuous functions vanishing at infinity on the quantum  $SU(1,1)$  group is studied, which is left and right invariant in a weak sense with respect to a product defined using Wall functions. The Haar weight restricted to certain subalgebras are explicitly determined in terms of an infinitely supported Jackson integral and in terms of an infinitely supported Askey-Wilson type measure. For the evaluation the spectral analysis of explicit unbounded doubly infinite Jacobi matrices and some new summation formulas for basic hypergeometric series are needed. The spherical functions are calculated in terms of Askey-Wilson functions and big  $q$ -Jacobi functions. The corresponding spherical Fourier transforms are identified with special cases of the big  $q$ -Jacobi function transform and of the Askey-Wilson function transform.

## 1. INTRODUCTION

The motivation for the study in this paper is twofold. On the one hand we are interested in the study of the simplest non-compact semisimple quantum group, namely the quantum  $SU(1,1)$  group, and in particular in its corresponding Haar functional. This quantum group is resisting any of the theories on locally compact quantum groups like e.g. [38]. On the other hand we are interested in special functions associated to quantum groups, and in particular in the so-called Askey-Wilson functions. Let us first say something on the second subject, which is our main concern.

A very general set of orthogonal polynomials in one variable is the set of Askey-Wilson polynomials introduced in 1985 in [5]. As the title of the memoir indicates, Askey-Wilson polynomials can be considered as  $q$ -analogues of the Jacobi polynomials which are orthogonal on  $[-1, 1]$  with respect to the beta integral  $(1-x)^\alpha(1+x)^\beta$ . The Jacobi polynomials are the polynomial solutions of the hypergeometric differential operator, whereas the

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1991 *Mathematics Subject Classification*. Primary 17B37, 33D55, 33D80, secondary 43A32, 43A90, 46L89, 47B15.

*Key words and phrases*. quantum  $SU(1,1)$  group, Haar functional, Wall functions, Jackson integral, Askey-Wilson integral,  $q$ -Jacobi functions, Askey-Wilson functions, spherical function, spherical Fourier transforms, spectral analysis, summation formulas.

Askey-Wilson polynomials are the polynomial solutions of a certain second-order difference operator. The Jacobi polynomials naturally arise as spherical functions on rank one compact Riemannian symmetric spaces. On the other hand, the spherical functions on non-compact rank one Riemannian symmetric spaces can be expressed in terms of Jacobi functions, which are non-polynomial eigenfunctions of the hypergeometric differential operator. The corresponding Fourier transforms are special cases of the Jacobi function transform in which the kernel is a Jacobi function. By now these Jacobi function transforms, containing as special cases the Fourier-cosine and Mehler-Fock transforms, are very well understood, see e.g. the survey paper [32] by Koornwinder and references therein. There are inversion formulas, as well as the appropriate analogues of the theorems of Plancherel, Parseval and Paley-Wiener. Furthermore, there are several different approaches to the study of the  $L^2$ -theory of the Jacobi function transform. One particular approach is by spectral analysis of the hypergeometric differential operator on a weighted  $L^2$ -space.

Although the Askey-Wilson functions are known, see [19], [49], [55], [56], as are all the solutions to the Askey-Wilson second order difference equation and their interrelations, it was not yet known what the appropriate Askey-Wilson function transform should be. The reason for this is our lack of understanding of the Hilbert space on which the Askey-Wilson difference operator has to be diagonalised. In case the Jacobi functions have an interpretation as spherical functions, the weighted  $L^2$ -space can be obtained by restricting the Haar measure to functions which behave as a character under the left and right action of a maximal compact subgroup. In this paper we show how the study of the Haar functional on the quantum  $SU(1,1)$  group can be used to find the right Hilbert spaces as weighted  $L^2$ -space and this is one of the main results of this paper. The actual analytic study of the Askey-Wilson function transform is done in another paper [29], and that of the appropriate limit case to the big  $q$ -Jacobi transform is done in [28]. This is for two reasons. The quantum group theoretic approach does not lead to a rigorous proof, and secondly the interpretation on the quantum group only holds for a restricted set of the parameters involved. It has to be noted that the Askey-Wilson function transform that occurs in this paper is different from the orthogonality relations introduced by Suslov [55], [56], see also [9] for the more extensively studied little  $q$ -Jacobi case, which is analogous to Fourier(-Bessel) series.

The motivation for the method we employ is the relation between special functions and the theory of group (and quantum group) representations. The Jacobi polynomials occur as matrix coefficients of irreducible unitary representations of the compact Lie group  $SU(2)$  and the Jacobi functions arise as matrix coefficients of irreducible unitary representations of the non-compact Lie group  $SU(1,1) \cong SL(2, \mathbb{R})$ , see [60], [61], [32]. These groups are both real forms of the same complex Lie group  $SL(2, \mathbb{C})$ . In the theory of quantum groups, the quantum analogue of the complex case  $SL(2, \mathbb{C})$  is much studied, as is the quantum analogue of the compact  $SU(2)$ , see e.g. [10]. One of the first indications that the relation between quantum groups and special functions is very strong, is the interpretation of the little  $q$ -Jacobi polynomials on the quantum  $SU(2)$  group as matrix elements on which the subgroup  $K = S(U(1) \times U(1)) \cong U(1)$  acts by a character. Since we can view little  $q$ -Jacobi polynomials as limiting cases of the Askey-Wilson polynomials, this is a first step. The breakthrough has come with Koornwinder's paper [35] in which he gives an infinitesimal

characterisation of quantum subgroups. This gives a one-parameter family of quantum subgroups, which we denote by  $K_t$ . The subgroups  $K_t$  and  $K_s$  are formally conjugated, see [50, §4]. Then the matrix elements on which  $K_s$ , respectively  $K_t$ , acts by a character from the left, respectively right, can be expressed in terms of Askey-Wilson polynomials. The in-between case of the big  $q$ -Jacobi polynomials can be obtained in a similar way. As a corollary to these results we get from the Schur orthogonality relations an explicit expression for the Haar functional on certain commutative subalgebras in terms of the Askey-Wilson orthogonality measure. For the spherical case, i.e. the matrix elements that are left and right invariant under  $K$ , we state this in the following table for the quantum  $SU(2)$  group. The spherical case is the important case to calculate.

<i>subgroups</i>	<i>Haar functional</i>	<i>spherical functions</i>
$(K, K)$	Jackson integral on $[0, 1]$	little $q$ -Legendre polynomials
$(K, K_t)$	Jackson integral on $[-t, 1]$	big $q$ -Legendre polynomials
$(K_s, K_t)$	Askey-Wilson integral	2-parameter Askey-Wilson polynomials

TABLE 1.1. Spherical functions for  $SU_q(2)$ 

For the quantum  $SU(1,1)$  group the matrix elements that behave nicely under the action of the subgroup  $K = S(U(1) \times U(1)) \cong U(1)$  have been calculated explicitly by Masuda et al. [41] and Vaksman and Korogodskii [57]. These can be expressed in terms of little  $q$ -Jacobi functions, and the Haar functional is also known in terms of a Jackson integral on  $[0, \infty)$ , see [21], [22], [42], [57]. This gives rise to the first line in the following table.

<i>subgroups</i>	<i>Haar functional</i>	<i>spherical functions</i>
$(K, K)$	Jackson integral on $[0, \infty)$	little $q$ -Legendre functions
$(K, K_t)$	Jackson integral on $[-t, \infty)$	big $q$ -Legendre functions
$(K_s, K_t)$	Askey-Wilson type integral	2-parameter Askey-Wilson functions

TABLE 1.2. Spherical functions for  $SU_q(1,1)$ 

The purpose of this paper is to prove the last two lines of Table 1.2. The proof of the explicit expression for the Haar functional in the last two cases of Table 1.2 is the main result of this paper. Koornwinder's proof for the cases in the compact setting listed in Table 1.1 cannot be used here, but the alternative proof using spectral theory and bilinear generating functions given in [31] can be generalised to the quantum  $SU(1,1)$  group. For this we give an expression for a Haar functional on the quantum  $SU(1,1)$  group in terms of representations of the quantised function algebra. In the last section we then formally show how the big  $q$ -Jacobi function transform and the Askey-Wilson function transform can be interpreted as Fourier transforms on the quantum  $SU(1,1)$  group. Because the Fourier transforms associated with  $SU(1,1)$  are special case of the Jacobi function transforms, see [32], we view the big  $q$ -Jacobi and Askey-Wilson function transform as  $q$ -analogues of the Jacobi function transform. The complete analytic study of the big  $q$ -Jacobi and Askey-Wilson function transform is developed in [28] and [29].

We expect that the Askey-Wilson function transform will play a central role in the theory of integral transforms with basic hypergeometric kernels. Indeed, in the polynomial setting, the Askey-Wilson polynomials have had a tremendous impact in the theory of basic hypergeometric orthogonal polynomials. Furthermore, the Jacobi function transform, which we consider as the classical counterpart of the Askey-Wilson function transform, has turned out to be an important integral transform in the theory of special functions and its applications. In particular, due to the quantum group theoretic interpretation of Askey-Wilson functions in this paper, we may expect that appropriate non-polynomial analogues of the results on Askey-Wilson polynomials in e.g. [13], [14], [27], [34], [44] exist.

The theory of locally compact quantum groups has not yet reached the state of maturity, but see Kustermans and Vaes [37], [38]. The quantum  $SU(1, 1)$  group does not fit into these theories because it lacks a good definition of the comultiplication defined on the  $C^*$ -algebra level, see [62], and without a comultiplication it is not possible to speak of left- and right invariance of a functional. In §2 we propose a weak version of the comultiplication, in the sense that we define a product for linear functionals in terms of Wall functions. Then we can show that our definition of the Haar functional is indeed left- and right invariant with respect to this weak version of the comultiplication.

Let us now turn to the contents of this paper. In §2 we introduce the quantum  $SU(1, 1)$  group and a corresponding  $C^*$ -algebra, which can be seen as the algebra of continuous functions vanishing at infinity on the quantum  $SU(1, 1)$  group. We work here with a faithful representation of the  $C^*$ -algebra, and we can introduce a weight, i.e. an unbounded functional, that is left and right invariant in the weak sense. This analogue of the Haar functional is an integral of weighted traces in irreducible representations of the  $C^*$ -algebra. In §3 we recall some facts on the algebraic level, both for the quantised algebra of polynomials on  $SU(1, 1)$  and for the quantised universal enveloping algebra. This part is mainly intended for notational purposes and for stating properties that are needed in the sequel. In §4 we prove the statement for the Haar functional in the second line of Table 1.2. This is done by a spectral analysis of a three-term recurrence operator in  $\ell^2(\mathbb{Z})$  previously studied in [11]. In §5 we prove the statement for the Haar functional in the third line of Table 1.2. A spectral analysis of a five-term recurrence operator in  $\ell^2(\mathbb{Z})$  is needed, and we can do it by factorising it as the product of two three-term recurrence operators. The factorisation is motivated by factorisation results on the quantum group level. At a certain point, Lemma 5.5, we require a highly non-trivial summation formula for basic hypergeometric series, and the derivation by Mizan Rahman is given in Appendix B. The result is an Askey-Wilson type measure with absolutely continuous part supported on  $[-1, 1]$  plus an infinite set of discrete mass points tending to infinity. In §6 we mainly study the spherical Fourier transforms on the quantum  $SU(1, 1)$  group. In this section we have to take a number of formal steps. We show that the radial part of the Casimir operator corresponds to a 2-parameter Askey-Wilson difference operator and we calculate the spherical functions in terms of very-well-poised  ${}_8\varphi_7$ -series. By the results of [29] we can invert the spherical Fourier transform and we see that the Plancherel measure is supported on the principal unitary series representations and an infinite discrete subset of the strange series representations. Finally, Appendix A contains the spectral analysis of a three-term operator on  $\ell^2(\mathbb{Z})$  extending the results of Kakehi [21] and Appendix B,

by Mizan Rahman, contains a number of summation formulas needed in the paper.

*Notation.* We use  $\mathbb{N} = \{1, 2, \dots\}$ ,  $\mathbb{Z}_{\geq 0} = \{0, 1, \dots\}$ , and  $q$  is a fixed number with  $0 < q < 1$ . For basic hypergeometric series we use Gasper and Rahman [17]. So for  $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$  we use the notation  $(a; q)_k = \prod_{i=0}^{k-1} (1 - aq^i)$  for  $q$ -shifted factorials, and also  $(a_1, \dots, a_r; q)_k = \prod_{i=1}^r (a_i; q)_k$ . The basic hypergeometric series is defined by

$${}_r\varphi_s \left( \begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, z \right) = \sum_{k=0}^{\infty} \frac{(a_1, \dots, a_r; q)_k z^k}{(q, b_1, \dots, b_s; q)_k} \left( (-1)^k q^{\frac{1}{2}k(k-1)} \right)^{s+1-r},$$

whenever it is well-defined. The series is balanced if  $r = s + 1$ ,  $b_1 \dots b_s = qa_1 \dots a_{s+1}$  and  $z = q$ . The series is called very-well-poised if  $r = s + 1$ ,  $qa_1 = a_2b_1 = a_3b_2 = \dots = a_{s+1}b_s$  and  $a_2 = q\sqrt{a_1}$ ,  $a_3 = -q\sqrt{a_1}$ . For the very-well-poised series we use the notation

$$\begin{aligned} {}_{s+1}W_s(a_1; a_4, \dots, a_{s+1}; q, z) &= {}_{s+1}\varphi_s \left( \begin{matrix} a_1, qa_1^{\frac{1}{2}}, -qa_1^{\frac{1}{2}}, a_4, \dots, a_{s+1} \\ a_1^{\frac{1}{2}}, -a_1^{\frac{1}{2}}, qa_1/a_4, \dots, qa_1/a_{s+1} \end{matrix}; q, z \right) \\ &= \sum_{k=0}^{\infty} \frac{1 - a_1q^{2k}}{1 - a_1} \frac{(a_1, a_4, \dots, a_{s+1}; q)_k z^k}{(q, qa_1/a_4, \dots, qa_1/a_{s+1}; q)_k}. \end{aligned}$$

*Acknowledgement.* We are grateful to Mizan Rahman for his crucial help, which resulted in Appendix B. We thank Tom Koornwinder for giving us access to unpublished notes, that have served as the basis for Appendix A, and for creating the stimulating environment at the Universiteit van Amsterdam, where both authors were based when this project began. The second author is supported by a NWO-Talent stipendium of the Netherlands Organization for Scientific Research (NWO). Part of the research was done while the second author was supported by the EC TMR network ‘‘Algebraic Lie Representations’’, grant no. ERB FMRX-CT97-0100. We thank Alfons Van Daele, Johan Kustermans, Erik Thomas and Hjalmar Rosengren for useful discussions.

## 2. THE QUANTUM $SU(1,1)$ GROUP

The quantum  $SU(1,1)$  group is introduced as a Hopf  $*$ -algebra. In §2.1 we describe its irreducible  $*$ -representations in terms of unbounded operators. In §2.2 we use these representations to define a  $C^*$ -algebra, which we regard as the algebra of continuous functions on the quantum  $SU(1,1)$  group which tend to zero at infinity, and we define a Haar weight on the  $C^*$ -algebra. The Haar weight and the  $C^*$ -algebra are the same as previously introduced for the quantum group of plane motions by Woronowicz [62] and also studied by Baaj [6], Quaegebeur and Verding [48], Verding [59]. We define a new product on certain linear functionals in terms of Wall functions, which reflect the comultiplication of the quantum  $SU(1,1)$  group. For this product we show that the Haar weight is right and left invariant.

**2.1 Representations of  $A_q(SU(1,1))$ .** We first recall some generalities on the quantum  $SL(2, \mathbb{C})$  group and a non-compact real form, the quantum  $SU(1,1)$  group, see e.g. Chari and Pressley [10] or any other textbook on quantum groups. Let  $A_q(SL(2, \mathbb{C}))$  be the unital algebra over  $\mathbb{C}$  generated by  $\alpha, \beta, \gamma$  and  $\delta$  satisfying

$$(2.1) \quad \begin{aligned} \alpha\beta &= q\beta\alpha, & \alpha\gamma &= q\gamma\alpha, & \beta\delta &= q\delta\beta, & \gamma\delta &= q\delta\gamma, \\ \beta\gamma &= \gamma\beta, & \alpha\delta - q\beta\gamma &= \delta\alpha - q^{-1}\beta\gamma = 1, \end{aligned}$$

where 1 denotes the unit of  $A_q(SL(2, \mathbb{C}))$  and  $0 < q < 1$ . A linear basis for this algebra is given by  $\{\alpha^k \beta^l \gamma^m \mid k, l, m \in \mathbb{Z}_{\geq 0}\} \cup \{\delta^k \beta^l \gamma^m \mid k \in \mathbb{N}, l, m \in \mathbb{Z}_{\geq 0}\}$ . This is a Hopf-algebra with comultiplication  $\Delta: A_q(SL(2, \mathbb{C})) \rightarrow A_q(SL(2, \mathbb{C})) \otimes A_q(SL(2, \mathbb{C}))$ , which is an algebra homomorphism, given by

$$\begin{aligned} \Delta(\alpha) &= \alpha \otimes \alpha + \beta \otimes \gamma, & \Delta(\beta) &= \alpha \otimes \beta + \beta \otimes \delta, \\ \Delta(\gamma) &= \gamma \otimes \alpha + \delta \otimes \gamma, & \Delta(\delta) &= \delta \otimes \delta + \gamma \otimes \beta, \end{aligned}$$

and counit  $\varepsilon: A_q(SL(2, \mathbb{C})) \rightarrow \mathbb{C}$ , which is an algebra homomorphism, given by  $\varepsilon(\alpha) = \varepsilon(\delta) = 1, \varepsilon(\beta) = \varepsilon(\gamma) = 0$ . There is also an antipode  $S: A_q(SL(2, \mathbb{C})) \rightarrow A_q(SL(2, \mathbb{C}))$ , which is an antimultiplicative linear mapping given on the generators by  $S(\alpha) = \delta, S(\beta) = -q^{-1}\beta, S(\gamma) = -q\gamma$  and  $S(\delta) = \alpha$ . We say that a linear functional  $h: A_q(SL(2, \mathbb{C})) \rightarrow \mathbb{C}$  is right invariant, respectively left invariant, if  $(h \otimes \text{id})\Delta(a) = h(a)1$ , respectively  $(\text{id} \otimes h)\Delta(a) = h(a)1$ , in  $A_q(SL(2, \mathbb{C}))$ . So  $h$  is a right invariant Haar functional if and only if  $h \star \omega = \omega(1)h$  for any linear functional  $\omega: A_q(SL(2, \mathbb{C})) \rightarrow \mathbb{C}$ , where the product of two linear functionals  $\omega, \omega'$  is defined by  $\omega \star \omega' = (\omega \otimes \omega') \circ \Delta$ .

With  $A_q(SU(1,1))$  we denote the  $*$ -algebra which is  $A_q(SL(2, \mathbb{C}))$  as an algebra with  $*$  given by  $\alpha^* = \delta, \beta^* = q\gamma, \gamma^* = q^{-1}\beta, \delta^* = \alpha$ . So  $A_q(SU(1,1))$  is the  $*$ -algebra generated by  $\alpha$  and  $\gamma$  subject to the relations

$$(2.2) \quad \alpha\gamma = q\gamma\alpha, \quad \alpha\gamma^* = q\gamma^*\alpha, \quad \gamma\gamma^* = \gamma^*\gamma, \quad \alpha\alpha^* - q^2\gamma^*\gamma = 1 = \alpha^*\alpha - \gamma\gamma^*.$$

This is in fact a Hopf  $*$ -algebra, implying that  $\Delta$  and  $\varepsilon$  are  $*$ -homomorphisms and  $S \circ *$  is an involution. In particular,

$$(2.3) \quad \Delta(\alpha) = \alpha \otimes \alpha + q\gamma^* \otimes \gamma, \quad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

We can represent the  $*$ -algebra  $A_q(SU(1,1))$  by unbounded operators in the Hilbert space  $\ell^2(\mathbb{Z})$  with standard orthonormal basis  $\{e_k \mid k \in \mathbb{Z}\}$ . Since the representation involves unbounded operators we have to be cautious. We stick to the conventions of Schmüdgen [52, Def. 8.1.9]: given a dense linear subspace  $\mathcal{D}$  of a Hilbert space  $\mathcal{H}$ , a mapping  $\pi$  of a unital  $*$ -algebra  $A$  into the set of linear operators defined on  $\mathcal{D}$  is a  $*$ -representation of  $A$  if

- (i)  $\pi(c_1 a_1 + c_2 a_2)v = c_1 \pi(a_1)v + c_2 \pi(a_2)v$  and  $\pi(1)v = v$  for all  $a_i \in A, c_i \in \mathbb{C} (i = 1, 2)$  and all  $v \in \mathcal{D}$ ,
- (ii)  $\pi(b)\mathcal{D} \subseteq \mathcal{D}$  and  $\pi(ab)v = \pi(a)\pi(b)v$  for all  $a, b \in A$  and all  $v \in \mathcal{D}$ ,
- (iii)  $\langle \pi(a)v, w \rangle = \langle v, \pi(a^*)w \rangle$  for all  $a \in A$  and all  $v, w \in \mathcal{D}$ .

Note that (iii) states that the domain of the adjoint  $\pi(a)^*$  contains  $\mathcal{D}$  for any  $a \in A$  and  $\pi(a)^*|_{\mathcal{D}} = \pi(a^*)$ , so that  $\pi(a)$  is closeable. See also Woronowicz [62, §4].

By  $\mathcal{D}(\mathbb{Z})$  we denote the dense subspace of  $\ell^2(\mathbb{Z})$  consisting of finite linear combinations of the standard basis vectors  $e_k$ ,  $k \in \mathbb{Z}$ .

**Proposition 2.1.** (i) *Let  $\lambda \in \mathbb{C} \setminus \{0\}$ . There exists a unique  $*$ -representation  $\pi_\lambda$  of  $A_q(SU(1,1))$  acting on  $\ell^2(\mathbb{Z})$  with common domain  $\mathcal{D}(\mathbb{Z})$ , such that*

$$\begin{aligned} \pi_\lambda(\alpha) e_k &= \sqrt{1 + |\lambda|^2 q^{-2k}} e_{k+1}, & \pi_\lambda(\gamma) e_k &= \lambda q^{-k} e_k, \\ \pi_\lambda(\alpha^*) e_k &= \sqrt{1 + |\lambda|^2 q^{2-2k}} e_{k-1}, & \pi_\lambda(\gamma^*) e_k &= \bar{\lambda} q^{-k} e_k. \end{aligned}$$

(ii) *The  $*$ -representation  $\pi_\lambda$  is irreducible and for  $\lambda, \mu \in R = \{z \in \mathbb{C} \mid q < |z| \leq 1\}$  the  $*$ -representations  $\pi_\lambda$  and  $\pi_\mu$  are inequivalent for  $\lambda \neq \mu$ . This means that the space of intertwiners  $I_{\lambda,\mu} = \{T \in \mathcal{B}(\ell^2(\mathbb{Z})) \mid T(\mathcal{D}(\mathbb{Z})) \subseteq \mathcal{D}(\mathbb{Z}), T\pi_\mu(a)v = \pi_\lambda(a)Tv, \forall a \in A_q(SU(1,1)), \forall v \in \mathcal{D}(\mathbb{Z})\}$  equals  $\{0\}$  for  $\lambda \neq \mu$ ,  $\lambda, \mu \in R$  and equals  $\mathbb{C} \cdot 1$  for  $\lambda = \mu$ .*

*Remark.* These are precisely the representations described by Woronowicz [62, §4].

*Proof.* To prove (i) we observe that  $\pi_\lambda(a)$  preserves  $\mathcal{D}(\mathbb{Z})$  for  $a \in \{\alpha, \alpha^*, \gamma, \gamma^*\}$  so that we have compositions of these operators. It is a straightforward calculation to see that these operators satisfy the commutation relations (2.2). It follows that  $\pi_\lambda$  uniquely extends to an algebra homomorphism  $\pi_\lambda$  of  $A_q(SU(1,1))$  into the algebra of linear operators on  $\mathcal{D}(\mathbb{Z})$ . It remains to prove that  $\langle \pi_\lambda(a)v, w \rangle = \langle v, \pi_\lambda(a^*)w \rangle$  for all  $a \in A_q(SU(1,1))$  and  $v, w \in \mathcal{D}(\mathbb{Z})$ , which follows from checking it for the generators  $a = \alpha$  and  $a = \gamma$ .

For (ii) we fix an intertwiner  $T \in I_{\lambda,\mu}$ . Then

$$(2.4) \quad \lambda q^{-k} T e_k = T(\pi_\lambda(\gamma) e_k) = \pi_\mu(\gamma)(T e_k), \quad k \in \mathbb{Z},$$

so  $T e_k = 0$  for all  $k \in \mathbb{Z}$  if  $\lambda \mu^{-1} \notin q^{\mathbb{Z}}$ . If  $\lambda, \mu \in R$ , then  $\lambda \mu^{-1} \notin q^{\mathbb{Z}} \Leftrightarrow \lambda \neq \mu$ . Hence,  $I_{\lambda,\mu} = \{0\}$  for  $\lambda, \mu \in R$  with  $\lambda \neq \mu$ .

If  $\lambda = \mu$ , then it follows from (2.4) that  $T e_k = c_k e_k$  for some  $c_k \in \mathbb{C}$ . Since  $T$  commutes with  $\pi_\lambda(\alpha)$ , it follows that  $c_k$  is independent of  $k$ , proving that  $I_{\lambda,\lambda} = \mathbb{C} \cdot 1$ .  $\square$

*Remark.* There is a canonical way to associate an adjoint representation  $\pi_\lambda^*$  to the representation  $\pi_\lambda$ , see [52, p. 202], by defining its common domain as the intersection of the domains of all adjoints,  $\mathcal{D}^* = \bigcap_{a \in A_q(SU(1,1))} \mathcal{D}(\pi_\lambda(a)^*)$ , and the action by  $\pi_\lambda^*(a) = \pi_\lambda(a^*)^*|_{\mathcal{D}^*}$ . For the domain  $\mathcal{D}(\mathbb{Z})$  we do not have self-adjointness of the representation  $\pi_\lambda$ . However, if we replace the common domain of  $\pi_\lambda$  by

$$\mathcal{S}(\mathbb{Z}) = \left\{ \sum_{k=-\infty}^{\infty} c_k e_k \in \ell^2(\mathbb{Z}) \mid \sum_{k=-\infty}^{\infty} q^{-2nk} |c_k|^2 < \infty, \forall n \in \mathbb{Z}_{\geq 0} \right\}$$

we do have  $\pi_\lambda^* = \pi_\lambda$ , i.e. the domains and operators are all the same. Indeed, observe that  $\mathcal{D}(\pi_\lambda(\gamma^n)^*) = \{\sum_{k=-\infty}^{\infty} c_k e_k \in \ell^2(\mathbb{Z}) \mid \sum_{k=-\infty}^{\infty} q^{-nk} c_k e_k \in \ell^2(\mathbb{Z})\}$  and hence  $\mathcal{D}^* \subseteq \mathcal{S}(\mathbb{Z})$ .

By [52, Prop. 8.1.2] this implies that the representation by unbounded operators is self-adjoint.

Having the irreducible  $*$ -representations of Proposition 2.1 we can form the direct integral  $*$ -representation  $\pi = (2\pi)^{-1} \int_0^{2\pi} \pi_{e^{i\phi}} d\phi$ , see [52, Def. 12.3.1], with its representation space  $(2\pi)^{-1} \int_0^{2\pi} \ell^2(\mathbb{Z}) d\phi \cong L^2(\mathbb{T}; \ell^2(\mathbb{Z}))$  equipped with the orthonormal basis  $e^{ix\phi} \otimes e_m$  for  $x, m \in \mathbb{Z}$ . The common domain is by definition

$$(2.5) \quad \mathcal{D}(L^2(\mathbb{T}; \ell^2(\mathbb{Z}))) = \{f \in L^2(\mathbb{T}; \ell^2(\mathbb{Z})) \mid f(e^{i\phi}) \in \mathcal{D}(\mathbb{Z}) \text{ a.e. and} \\ e^{i\phi} \mapsto \pi_{e^{i\phi}}(a)f(e^{i\phi}) \in L^2(\mathbb{T}; \ell^2(\mathbb{Z})) \forall a \in A_q(SU(1, 1))\}.$$

The last condition means in particular that  $(2\pi)^{-1} \int_0^{2\pi} \|\pi_{e^{i\phi}}(a)f(e^{i\phi})\|^2 d\phi < \infty$  for all  $a \in A_q(SU(1, 1))$ . In this case  $\mathcal{D}(L^2(\mathbb{T}; \ell^2(\mathbb{Z})))$  is dense in  $L^2(\mathbb{T}; \ell^2(\mathbb{Z}))$  since it contains finite linear combinations of the basis elements  $e^{ix\phi} \otimes e_m$ . The action of the generators of  $A_q(SU(1, 1))$  on the basis of  $L^2(\mathbb{T}; \ell^2(\mathbb{Z}))$  can be calculated explicitly from Proposition 2.1;

$$\begin{aligned} \pi(\gamma) e^{ix\phi} \otimes e_m &= q^{-m} e^{i(x+1)\phi} \otimes e_m, & \pi(\alpha) e^{ix\phi} \otimes e_m &= \sqrt{1+q^{-2m}} e^{ix\phi} \otimes e_{m+1}, \\ \pi(\gamma^*) e^{ix\phi} \otimes e_m &= q^{-m} e^{i(x-1)\phi} \otimes e_m, & \pi(\alpha^*) e^{ix\phi} \otimes e_m &= \sqrt{1+q^{2-2m}} e^{ix\phi} \otimes e_{m-1}. \end{aligned}$$

**Lemma 2.2.** *The direct integral representation  $\pi = (2\pi)^{-1} \int_0^{2\pi} \pi_{e^{i\phi}} d\phi$  is a faithful representation of the  $*$ -algebra  $A_q(SU(1, 1))$ , i.e.  $\pi(\xi)f = 0$  for all  $f \in \mathcal{D}(L^2(\mathbb{T}; \ell^2(\mathbb{Z})))$  implies  $\xi = 0$  in  $A_q(SU(1, 1))$ .*

*Proof.* The action of the monomial basis of  $A_q(SU(1, 1))$  under the representation  $\pi$  is given by

$$\begin{aligned} \pi(\alpha^r (\gamma^*)^s \gamma^t) e^{ip\theta} \otimes e_l &= q^{-l(s+t)} (-q^{-2l}; q^{-2})_r^{\frac{1}{2}} e^{i(p+t-s)\theta} \otimes e_{l+r}, \\ \pi((\alpha^*)^r (\gamma^*)^s \gamma^t) e^{ip\theta} \otimes e_l &= q^{-l(s+t)\theta} (-q^{2-2l}; q^2)_r^{\frac{1}{2}} e^{i(p+t-s)\theta} \otimes e_{l-r}. \end{aligned}$$

It easily follows that  $\pi$  is a faithful representation of  $A_q(SU(1, 1))$ .  $\square$

In the next subsection we give a coordinate-free realisation of the representation  $\pi$ .

**2.2 The Haar functional.** Let  $L^2(X, \mu)$  be the Hilbert space of square integrable functions on  $X = \mathbb{T} \times q^{\mathbb{Z}} \cup \{0\}$  with respect to the measure

$$\int f d\mu = \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} f(q^k e^{i\theta}) d\theta.$$

Then the map  $\psi: L^2(\mathbb{T}; \ell^2(\mathbb{Z})) \rightarrow L^2(X, \mu)$  given by

$$(2.6) \quad \psi: e^{ix\phi} \otimes e_m \mapsto \left( f_{x,m}: z \mapsto \delta_{|z|, q^{-m}} \left( \frac{z}{|z|} \right)^x \right)$$



is a unitary isomorphism. Observe that  $\psi\pi(\gamma)\psi^{-1} = M_z$ ,  $\psi\pi(\gamma^*)\psi^{-1} = M_{\bar{z}}$ ,  $\psi\pi(\alpha)\psi^{-1} = M_{\sqrt{1+q^2|z|^2}}T_q$  and  $\psi\pi(\alpha^*)\psi^{-1} = M_{\sqrt{1+|z|^2}}T_q^{-1}$ , where  $M_g$  denotes the operator of multiplication by the function  $g$  and  $T_q$  is the  $q$ -shift operator defined by  $(T_q f)(z) = f(qz)$ .

These formulas for the action of the generators of  $A_q(SU(1,1))$  under the faithful representation  $\psi\pi(\cdot)\psi^{-1}$  suggest the following formal definition for the  $C^*$ -algebra of continuous functions on the quantum  $SU(1,1)$  group which vanish at infinity: it is the  $C^*$ -subalgebra of  $\mathcal{B}(L^2(X, \mu))$  generated by  $M_g$ ,  $g \in C_0(X)$ , and the  $q$ -shifts  $T_{q^{\pm 1}}$ , where  $C_0(X)$  is the  $C^*$ -algebra consisting of continuous functions on  $X$  which vanish at infinity (here  $X$  inherits its topology from  $\mathbb{C}$ , and the  $C^*$ -norm is given by the supremum norm  $\|\cdot\|_\infty$ ). In other words, one replaces the unbounded action of the subalgebra  $\mathbb{C}[\gamma, \gamma^*] \subset A_q(SU(1,1))$  on  $L^2(X, \mu)$  by the bounded, regular action of  $C_0(X)$  on  $L^2(X, \mu)$ .

To make the construction rigorous we use the notion of a crossed product  $C^*$ -algebra, see [47, Ch. 7]. The crossed product needed here is the same as for the quantum group of plane motions, see Baaj [6], Woronowicz [62]. Let us recall the construction in this specific case. For  $k \in \mathbb{Z}$  we define the automorphisms  $\tau_k$  of  $C_0(X)$  by  $\tau_k(f) = (T_q)^k f$ . Then  $(C_0(X), \mathbb{Z}, \tau)$  is a  $C^*$ -dynamical system, see [47, §7.4].

Let  $\ell^1(\mathbb{Z}; C_0(X))$  be the  $\ell^1$ -functions  $f: \mathbb{Z} \rightarrow C_0(X)$  with respect to the norm  $\|f\|_1 = \sum_{n \in \mathbb{Z}} \|f_n\|_\infty$ . The subspace  $C_c(\mathbb{Z}; C_0(X)) = \{f: \mathbb{Z} \rightarrow C_0(X) \mid \#\text{supp}(f) < \infty\}$  is dense in  $\ell^1(\mathbb{Z}; C_0(X))$ . Furthermore,  $\ell^1(\mathbb{Z}; C_0(X))$  is a Banach  $*$ -algebra, with  $*$ -structure and multiplication given by

$$(2.7) \quad (f^*)_n = \tau_n((f_{-n})^*), \quad (fg)_n = \sum_{k=-\infty}^{\infty} f_k \tau_k(g_{n-k}).$$

The crossed  $C^*$ -product  $C_0(X) \times_\tau \mathbb{Z}$  is by definition the strong closure of  $\ell^1(\mathbb{Z}; C_0(X))$  under its universal representation, where the universal representation is the direct sum of the non-degenerate  $*$ -representations of  $\ell^1(\mathbb{Z}; C_0(X))$ , see [47, §7.6]. We regard  $C_0(X) \times_\tau \mathbb{Z}$  as the quantum analogue of the  $C^*$ -algebra consisting of continuous functions on  $SU(1,1)$  which vanish at infinity.

We interpret the faithful  $*$ -representation  $\pi$  of §2.1 as a non-degenerate representation of  $C_0(X) \times_\tau \mathbb{Z}$  in the following way. Let  $\tilde{\pi}$  be the regular representation of  $C_0(X)$  on  $L^2(X, \mu)$ , and let  $u: \mathbb{Z} \rightarrow \mathcal{B}(L^2(X, \mu))$  be the unitary representation defined by  $u_n f = (T_q)^n f$ . Then  $(\tilde{\pi}, u, L^2(X, \mu))$  is a covariant representation of the  $C^*$ -dynamical system  $(C_0(X), \mathbb{Z}, \tau)$ , i.e.  $\tilde{\pi}(\tau_n f) = u_n \tilde{\pi}(f) u_n^*$  for all  $f \in C_0(X)$  and  $n \in \mathbb{Z}$ . In the notation of [47], we get a non-degenerate representation  $\pi = \tilde{\pi} \times u$  of  $C_0(X) \times_\tau \mathbb{Z}$  on  $L^2(X, \mu)$ , which is defined on  $C_c(\mathbb{Z}; C_0(X))$  by  $\pi(f)g = \sum_{n \in \mathbb{Z}} \tilde{\pi}(f_n)(u_n g)$ . More explicitly, we have

$$(\pi(f)g)(z) = \sum_{n \in \mathbb{Z}} f_n(z)g(q^n z), \quad f \in C_c(\mathbb{Z}; C_0(X)), \quad g \in L^2(X, \mu), \quad x \in X.$$

The  $*$ -representations  $\pi_{e^{i\theta}}$  of §2.1 can also be considered as a covariant representation of  $C_0(X) \times_\tau \mathbb{Z}$ . Let the representation  $\tilde{\pi}_{e^{i\theta}}: C_0(X) \rightarrow \mathcal{B}(\ell^2(\mathbb{Z}))$  of the commutative  $C^*$ -algebra be defined by  $\tilde{\pi}_{e^{i\theta}}(f)e_l = f(q^{-l}e^{i\theta})e_l$ , and let the unitary representation  $u: \mathbb{Z} \rightarrow \mathcal{B}(\ell^2(\mathbb{Z}))$  be defined by  $u_n = U^n$ , where  $U: \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$ ,  $e_k \mapsto e_{k+1}$  is the shift operator. Then

this gives a covariant representation. The direct integral representation  $(2\pi)^{-1} \int_0^{2\pi} \pi_{e^{i\theta}} d\theta$  in  $(2\pi)^{-1} \int_0^{2\pi} \ell^2(\mathbb{Z}) d\theta \cong L^2(\mathbb{T}; \ell^2(\mathbb{Z}))$  is equivalent to  $\pi$  using  $\psi$  as in (2.6).

*Remark.* The matrix elements of  $\pi$  with respect to the orthonormal basis  $f_{k,l}$ ,  $k, l \in \mathbb{Z}$ , of  $L^2(X, \mu)$ , see (2.6), give the linear functionals

$$\omega_{s,l}^{r-k}(f) = \langle \pi(f) f_{k,l}, f_{r,s} \rangle_{L^2(X, \mu)} = \frac{1}{2\pi} \int_0^{2\pi} f_{s-l}(q^{-s} e^{i\theta}) e^{i(k-r)\theta} d\theta$$

for  $f \in C_c(\mathbb{Z}; C_0(X))$ , which can be uniquely extended by continuity to a linear functional on  $C_0(X) \times_{\tau} \mathbb{Z}$ . Note that

$$\omega_{k,l}^x(fg) = \sum_{r,y \in \mathbb{Z}} \omega_{k,r}^{x-y}(f) \omega_{r,l}^y(g).$$

**Proposition 2.3.**  $\pi$  is a faithful representation of  $C_0(X) \times_{\tau} \mathbb{Z}$ .

*Proof.* Recall that  $\tilde{\pi}$  is the regular representation of  $C_0(X)$  on  $L^2(X, \mu)$ . The regular representation of  $C_0(X) \times_{\tau} \mathbb{Z}$  induced by  $\tilde{\pi}$ , which we denote here by  $\rho$ , acts on  $\ell^2(\mathbb{Z}; L^2(X, \mu))$  by

$$(\rho(f)g)_n(z) = \sum_{m \in \mathbb{Z}} (\tilde{\pi}(\tau_{-n}(f_m))g_{n-m})(z) = \sum_{m \in \mathbb{Z}} f_m(q^{-n}z)g_{n-m}(z)$$

for  $f \in C_c(\mathbb{Z}; C_0(X))$ ,  $g \in \ell^2(\mathbb{Z}; L^2(X, \mu))$  and  $z \in X$ , see [47, Ch. 7]. Explicitly, the action of  $\rho$  in terms of the orthonormal basis  $g_{k,l,m} = \delta_k(\cdot) f_{l,m}$ ,  $k, l, m \in \mathbb{Z}$ , of  $\ell^2(\mathbb{Z}; L^2(X, \mu))$  is given by

$$\rho(f)g_{k,l,m} = \sum_{r,s} \left( \frac{1}{2\pi} \int_0^{2\pi} f_{r-k}(q^{-m-r} e^{i\theta}) e^{i(l-s)\theta} d\theta \right) g_{r,s,m}.$$

So the closure  $H_m$  of  $\text{span}\{g_{k,l,m} \mid k, l \in \mathbb{Z}\}$  is an invariant subspace for  $\rho$  and we have the orthogonal direct sum decomposition  $\ell^2(\mathbb{Z}; L^2(X, \mu)) = \oplus_{m \in \mathbb{Z}} H_m$ . From the explicit formulas for  $\pi$  and  $\rho$  with respect to the orthonormal basis  $f_{k,l}$ ,  $k, l \in \mathbb{Z}$ , of  $L^2(X, \mu)$ , respectively  $g_{k,l,m}$ ,  $k, l, m \in \mathbb{Z}$ , of  $\ell^2(\mathbb{Z}; L^2(X, \mu))$ , it follows that  $L^2(X, \mu) \rightarrow H_m$ ,  $f_{l,k} \mapsto g_{k-m,l,m}$  is a unitary intertwiner between  $\pi$  and  $\rho|_{H_m}$ , so that  $\rho \simeq \bigoplus_{m \in \mathbb{Z}} \pi$  as representations of  $C_0(X) \times_{\tau} \mathbb{Z}$ . By [47, Cor. 7.7.8] we know that  $\rho$  is a faithful representation of  $C_0(X) \times_{\tau} \mathbb{Z}$ , hence so is  $\pi$ .  $\square$

Recall that a *weight* on a  $C^*$ -algebra  $A$  is a function  $h: A^+ \rightarrow [0, \infty]$  satisfying (i)  $h(\lambda a) = \lambda h(a)$  for  $\lambda \geq 0$  and  $a \in A^+$ , (ii)  $h(a+b) = h(a) + h(b)$  for  $a, b \in A^+$ . The weight  $h$  is said to be densely defined if  $\{a \in A^+ \mid h(a) < \infty\}$  is dense in  $A^+$ . Furthermore, we say that  $h$  is lower semi-continuous if  $\{a \in A^+ \mid h(a) \leq \lambda\}$  is closed for any  $\lambda \geq 0$ , and that  $h$  is faithful if  $h(a^*a) = 0$  implies  $a = 0$  in  $A$ . A weight can be extended uniquely to  $N_h^* N_h = \{a^*b \mid a, b \in N_h\}$ , where  $N_h = \{a \in A \mid h(a^*a) < \infty\}$ . See Combes [12, §1], Pedersen [47, Ch. 5] for general information and for application of weights in quantum groups see Kustermans and Vaes [37], Quaegebeur and Verding [48], Verding [59].

Let  $h$  be a lower semi-continuous, densely defined weight on  $A$ . The GNS-construction for weights gives a Hilbert space  $H_h$  and a representation  $\sigma_h$  of  $A$  in  $H_h$  and a linear map  $\Lambda_h$  from  $N_h = \{f \in A \mid h(f^*f) < \infty\}$  onto a dense subspace of  $H_h$  satisfying

- (i)  $h(f^*g) = \langle \Lambda_h(g), \Lambda_h(f) \rangle$  for all  $f, g \in N_h$ ,
- (ii)  $\sigma_h(f)\Lambda_h(g) = \Lambda_h(fg)$  for all  $f \in A$  and all  $g \in N_h$ ,
- (iii) the representation  $\sigma_h$  is non-degenerate, i.e. the closure of the linear span of elements of the form  $\sigma_h(f)g$  for  $f \in A$  and  $g \in H_h$  equals  $H_h$ ,
- (iv) the map  $\Lambda_h: N_h \rightarrow H_h$  is closed.

Properties (i) and (ii) hold by the general GNS-construction of weights, see [12, §2], and properties (iii) and (iv) follow since  $h$  is lower semi-continuous, see [12, §2], [59, Prop. 2.1.11]. If  $h$  is faithful, we obtain  $H_h$  as the Hilbert space completion of  $N_h$  with respect to inner product  $\langle f, g \rangle = h(g^*f)$ .

The following theorem has been proved by Baaj [6, §4] in the setting of weights on von Neumann algebras and later in the  $C^*$ -algebra framework by Quaegebeur and Verding [48, §4], Verding [59, §3.2].

**Theorem 2.4.** *Let  $h = \sum_{k=-\infty}^{\infty} q^{-2k} \omega_{k,k}^0$ , then  $h$  is a densely defined, faithful, lower semi-continuous weight on  $C_0(X) \times_{\tau} \mathbb{Z}$ .*

*Remark 2.5.* Note that  $C_c(\mathbb{Z}; C_c(X)) \subset N_h$ , since for  $f \in C_c(\mathbb{Z}; C_c(X))$  we have  $f^*f \in C_c(\mathbb{Z}; C_c(X))$  so that  $\omega_{k,k}^0(f^*f) = 0$  for  $k$  sufficiently large. Also, the dense subspace  $C_c(\mathbb{Z}; C_c(X))$  of  $C_0(X) \times_{\tau} \mathbb{Z}$  is contained in  $N_h^* N_h$ , so that the Haar functional is well-defined on  $C_c(\mathbb{Z}; C_c(X))$ . Indeed, take  $g^{(m)} \in C_c(\mathbb{Z}; C_c(X))$ ,  $g_n^{(m)}(x) = \delta_{n,0} u^{(m)}(x)$ , where  $u^{(m)}$  is a compactly supported approximate unit in  $C_0(X)$  with support in  $|x| \leq q^{-m}$ . Then  $(g^{(m)})^* \in N_h$ , and  $g^{(m)}f = f$  for  $f \in C_c(\mathbb{Z}; C_c(X))$  and  $m$  sufficiently large.

*Remark.* We regard  $H_h$  in this paper as the  $q$ -analogue of the  $L^2$ -functions on  $SU(1,1)$  with respect to the Haar measure. It was shown by Baaj [6] and Quaegebeur and Verding [48] that  $H_h$  is isomorphic to  $\ell^2(\mathbb{Z}^3)$ . This corresponds nicely with the fact that  $SU(1,1)$  is a three-dimensional Lie group.

**2.3 Invariance of the Haar functional.** The weight  $h$  is left and right invariant when considered as a weight corresponding to the quantum group of plane motions, see Baaj [6, Thm. 4.2]. For this we have to have the comultiplication of the quantum group of plane motions on the  $C^*$ -algebra level, and this has been done by Woronowicz [62]. This seems not to be possible for the quantum  $SU(1,1)$  group, see Woronowicz [62, Thm. 4.1]. However we can introduce the comultiplication for the quantum  $SU(1,1)$  group on the  $C^*$ -algebra level in a weak form, and then the Haar weight is also left and right invariant. For this we first have to encode the comultiplication as in (2.3) in terms of a product for the matrix elements. See Baaj [6] for a similar procedure for the quantum group of plane motions.

**Lemma 2.6.** *For  $x, m, k \in \mathbb{Z}$  define the normalised Wall function by*

$$f_m^x(k) = \frac{(-1)^k q^{(k-m)(1+x)} (-q^{2-2k}; q^2)_{\infty}^{\frac{1}{2}} (q^{2+2x}; q^2)_{\infty}}{(-q^{2-2m}, -q^{2-2k+2x}; q^2)_{\infty}^{\frac{1}{2}} (q^2; q^2)_{\infty}} {}_1\varphi_1 \left( \begin{matrix} -q^{2+2x-2k} \\ q^{2+2x} \end{matrix}; q^2, q^{2+2k-2m} \right).$$

The product of linear functionals  $\omega_{k,l}^x$  given by

$$\omega_{k,l}^x \star \omega_{r,s}^y = \delta_{l-k-y, s-r+x} (-1)^{l-k-y} \sum_{n=-\infty}^{\infty} f_{n+l-k-y}^{s-l}(s) f_n^{r-k}(r) \omega_{n, n+l-k-y}^{r-s+l-k}$$

is well-defined as a linear functional on  $C_0(X) \times_{\tau} \mathbb{Z}$ .

*Remark 2.7.* In order to motivate the definition of Lemma 2.6 let us first consider the case of the compact quantum  $SU(2)$  group. The analogue of the representations  $\pi_{e^{i\theta}}$  are in terms of bounded operators on  $\ell^2(\mathbb{Z}_{\geq 0})$  and the tensor product decomposition  $\pi_{e^{i\theta}} \otimes \pi_{e^{i\psi}} \cong (2\pi)^{-1} \int_0^{2\pi} \pi_{e^{i\phi}} d\phi$  holds, see [33], [53]. Moreover, the Clebsch-Gordan coefficients are explicitly given in terms of Wall polynomials, see [33], and the Clebsch-Gordan coefficients are determined by a spectral analysis of the compact operator  $(\pi_{e^{i\theta}} \otimes \pi_{e^{i\psi}}) \Delta(\gamma^* \gamma)$ . This is done by interpreting this operator as a three-term recurrence operator, i.e. as a Jacobi matrix, corresponding to the Wall polynomials. The Clebsch-Gordan coefficients then determine the product  $\omega \star \omega' = (\omega \otimes \omega') \circ \Delta$  of the matrix elements  $\omega, \omega'$ , see also [6, Prop. 4.3] for the quantum group of plane motions. For the quantum  $SU(1,1)$  group we can formally follow the same method using the representations  $\pi_{e^{i\theta}}$  as in §2.1. In this case we have from Proposition 2.1 and (2.3)

$$\begin{aligned} (\pi_{e^{i\theta}} \otimes \pi_{e^{i\psi}}) \Delta(\gamma^* \gamma) e_k \otimes e_l &= (q^{-2k}(1+q^{-2l}) + q^{-2l}(1+q^{2-2k})) e_k \otimes e_l \\ &\quad + e^{i(\theta-\psi)} q^{-k-l-1} ((1+q^{-2k})(1+q^{-2l}))^{\frac{1}{2}} e_{k+1} \otimes e_{l+1} \\ &\quad + e^{i(\psi-\theta)} q^{-k-l+1} ((1+q^{2-2k})(1+q^{2-2l}))^{\frac{1}{2}} e_{k-1} \otimes e_{l-1}. \end{aligned}$$

Hence, the unbounded operator  $(\pi_{e^{i\theta}} \otimes \pi_{e^{i\psi}}) \Delta(\gamma^* \gamma)$  leaves the subspace  $(\mathcal{D}(\mathbb{Z}) \times \mathcal{D}(\mathbb{Z})) \cap H_x$  invariant, where  $H_x$  is the closure of  $\text{span}\{e_{k-x} \otimes e_k \mid k \in \mathbb{Z}\}$ , so that  $H_x \cong \ell^2(\mathbb{Z})$  for any  $x \in \mathbb{Z}$ . Restricting the unbounded symmetric operator  $(\pi_{e^{i\theta}} \otimes \pi_{e^{i\psi}}) \Delta(\gamma^* \gamma)$  to  $\mathcal{D}(H_x)$  gives an unbounded symmetric three-term recurrence operator in  $\ell^2(\mathbb{Z})$ , i.e. a doubly infinite Jacobi matrix, of the form

$$\begin{aligned} L e_k &= a_k e_{k+1} + b_k e_k + a_{k-1} e_{k-1}, \quad a_k > 0, \quad b_k \in \mathbb{R}, \\ a_k &= q^{(x-1)-2k} ((1+q^{2(x-k)})(1+q^{-2k}))^{\frac{1}{2}}, \quad b_k = q^{-2k}(1+q^{2x}) + q^{2x-4k}(1+q^2), \end{aligned}$$

where  $e_k$  is given by  $e^{ik(\psi-\theta)} e_{k-x} \otimes e_k$ . This operator fits into the general framework as studied in Appendix A. Since we have solutions in terms of Wall functions, see Gupta et al. [18, §5], as well as the asymptotically well-behaved solutions we can work out the details. We find that the elements

$$F_m^x = \sum_{k \in \mathbb{Z}} (-1)^m e^{ix\psi} e^{i(k-m)(\theta-\psi)} f_m^x(k) e_{k-x} \otimes e_k \in \ell^2(\mathbb{Z} \times \mathbb{Z})$$

are eigenvectors of  $(\pi_{e^{i\theta}} \otimes \pi_{e^{i\psi}}) \Delta(\gamma^* \gamma)$  in  $H_x$  for the eigenvalue  $q^{-2m}$ ,  $m \in \mathbb{Z}$ . The vectors are contained in the domain of the adjoint  $L^*$ . Using contiguous relations we can formally

show that the action of  $(\pi_{e^{i\theta}} \otimes \pi_{e^{i\psi}})\Delta(a)$ ,  $a \in A_q(SU(1,1))$ , on  $F_m^x$  is the same as the action of  $\pi$  on the orthonormal basis  $\{f_{x,m}\}$  of  $L^2(X, \mu)$  by identifying  $F_m^x$  with  $f_{x,m}$ , see (2.6). However,  $L$  has deficiency indices  $(1,1)$  and there is no self-adjoint extension of  $L$  possible such that these eigenvectors are contained in the domain of the self-adjoint extension of  $L$ , so the  $\{F_m^x\}$  are not orthogonal, see [11]. Note that this observation corresponds to the no-go theorem of Woronowicz [62, Thm. 4.1]. Ignoring this problem and regarding the  $*$ -representations  $(\pi_{e^{i\theta}} \otimes \pi_{e^{i\psi}})\Delta$  in  $\ell^2(\mathbb{Z} \times \mathbb{Z})$  and  $\pi$  in  $L^2(X, \mu)$  equivalent and using the identity, see [11, (4.3)],

$$(2.8) \quad e_l \otimes e_s = \sum_{m \in \mathbb{Z}} (-1)^m e^{i\psi(l-s)} e^{i(m-s)(\theta-\psi)} f_m^{s-l}(s) F_m^{s-l},$$

we formally can rewrite

$$(\omega_{k,l}^x \otimes \omega_{r,s}^y) \circ \Delta = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} e^{-ix\theta} e^{-iy\psi} \langle (\pi_{e^{i\theta}} \otimes \pi_{e^{i\psi}})\Delta(\cdot) e_l \otimes e_s, e_k \otimes e_r \rangle d\theta d\psi$$

as the linear combination of Lemma 2.6 using (2.8). So we cannot make this method rigorous, but the results of [28, §2] suggest that we should consider  $L$  on a bigger Hilbert space. Finally, we note that for the quantum group of plane motions the intertwining operator consisting of the Clebsch-Gordan coefficients can be used to find a multiplicative unitary, see Baaj [6, §4]. For the quantum  $SU(1,1)$  we only obtain a partial isometry and we do not expect a multiplicative unitary from this construction.

*Proof of Lemma 2.6.* Since  $|\omega_{k,l}^x(f)| \leq \|f\|$  for any  $f \in C_0(X) \times_{\tau} \mathbb{Z}$ , it suffices to show that

$$\sum_{n=-\infty}^{\infty} f_{n+l-k-y}^{s-l}(s) f_n^{r-k}(r)$$

is absolutely convergent. Since  $\{f_m^x(k)\}_{m \in \mathbb{Z}} \in \ell^p(\mathbb{Z})$ ,  $1 \leq p \leq \infty$ , see the next lemma and the remark following it, this follows immediately.  $\square$

**Lemma 2.8.** *For  $x, m, k \in \mathbb{Z}$  the Wall functions of Lemma 2.6 satisfy  $f_m^{-x}(k) = f_m^x(k+x)$ . Furthermore, for  $x \geq 0$ ,*

$$|f_m^x(k)| \leq \frac{(-q^{2-2k}, -q^{2-2m}; q^2)_{\infty}^{\frac{1}{2}} (-q^{2+2x}; q^2)_{\infty}}{(-q^{2-2k+2x}; q^2)_{\infty}^{\frac{1}{2}} (q^2; q^2)_{\infty}} \begin{cases} q^{(k-m)(1+x)}, & m \leq k, \\ q^{(m-k)(1+x)} q^{(m-k)(m-k-1)}, & m \geq k. \end{cases}$$

*Remark.* The  $\ell^p$ -behaviour of  $\{f_m^x(k)\}_{m=-\infty}^{\infty}$  follows from Lemma 2.8, since for fixed  $x$  and  $k$  we see  $f_m^x(k) = \mathcal{O}(q^{-m(1+|x|)})$  as  $m \rightarrow -\infty$  and  $f_m^x(k) = \mathcal{O}(q^{\frac{1}{2}m(m-1)} q^{m(1+x-2k)})$  as  $m \rightarrow \infty$  using the theta product identity

$$(2.9) \quad (aq^k, q^{1-k}/a; q)_{\infty} = (-a)^{-k} q^{-\frac{1}{2}k(k-1)} (a, q/a; q)_{\infty}, \quad a \in \mathbb{C} \setminus \{0\}, \quad k \in \mathbb{Z}.$$

*Proof.* First observe

$$(2.10) \quad \sum_{p=0}^{\infty} \frac{(q^{1-n}; q)_{\infty} c_p}{(q, q^{1-n}; q)_p} = \sum_{p=n}^{\infty} \frac{(q^{1-n+p}; q)_{\infty} c_p}{(q; q)_p} = \sum_{p=0}^{\infty} \frac{(q^{1+n}; q)_{\infty} c_{p+n}}{(q, q^{1+n}; q)_p}$$

for  $n \in \mathbb{Z}$  provided that the sums are absolutely convergent. Applying this with  $n = -x$  gives  $f_m^{-x}(k) = f_m^x(k+x)$ .

In order to estimate  $f_m^x(k)$ ,  $x \geq 0$ , we use the following limit transition of Heine's formula, see [17, (1.4.5)],

$$(2.11) \quad (c; q)_\infty {}_1\varphi_1(a; c; q, z) = (z; q)_\infty {}_1\varphi_1(az/c; z; q, c),$$

to rewrite the  ${}_1\varphi_1$ -series in the definition of the Wall function as follows

$$(2.12) \quad (q^{2+2x}; q^2)_\infty {}_1\varphi_1\left(\frac{-q^{2+2x-2k}}{q^{2+2x}}; q^2, q^{2+2k-2m}\right) = \\ (q^{2+2k-2m}; q^2)_\infty {}_1\varphi_1\left(\frac{-q^{2-2m}}{q^{2+2k-2m}}; q^2, q^{2+2x}\right) = \\ \sum_{l=0}^{\infty} (q^{2+2k-2m+2l}; q^2)_\infty \frac{(-q^{2-2m}; q^2)_l}{(q^2; q^2)_l} (-1)^l q^{l(l-1)} q^{l(2+2x)}.$$

Note that for  $k-m \geq 0$  the sum starts at  $l=0$ , but for  $k-m \leq 0$  the sum actually starts at  $l=m-k$ , cf. (2.10). In case  $k \geq m$  we estimate the right hand side of (2.12) termwise to find the bound

$$(-q^{2-2m}; q^2)_\infty \sum_{l=0}^{\infty} \frac{q^{l(l-1)}}{(q^2; q^2)_l} q^{l(2+2x)} = (-q^{2-2m}, -q^{2+2x}; q^2)_\infty$$

by [17, (1.3.16)]. Combining this with the definition of  $f_m^x(k)$  gives the desired estimate in this case.

In case  $m \geq k$  we rewrite the sum on the right hand side of (2.12), by introducing  $l = n + m - k$ , as, cf. (2.10),

$$(-1)^{m-k} q^{(m-k)(m-k-1)} q^{(m-k)(2+2x)} (-q^{2-2m}; q^2)_{m-k} \\ \times \sum_{n=0}^{\infty} (q^{2+2m-2k+2n}; q^2)_\infty \frac{(-q^{2-2k}; q^2)_n}{(q^2; q^2)_n} (-1)^n q^{n(n-1)} q^{2n(1+x+m-k)}$$

and the sum is estimated by  $(-q^{2-2k}, -q^{2+2x+2m-2k}; q^2)_\infty \leq (-q^{2-2k}, -q^{2+2x}; q^2)_\infty$  in the same way. This gives the result for the case  $m \geq k$ .  $\square$

In  $(C_0(X) \times_\tau \mathbb{Z})^*$  the set  $\{\omega_{k,l}^x\}$  is linearly independent. This follows by applying it to the elements  $g_{k,l,m} \in C_c(\mathbb{Z}; C_c(X))$ , see Proposition 2.3. So we have a well-defined product on linear functionals from  $B \subset (C_0(X) \times_\tau \mathbb{Z})^*$ ,  $B$  being the space of finite linear combinations of the functionals  $\omega_{k,l}^x$ . For  $\omega \in B$  we extend the definition of the product to  $\omega \star h$  by requiring that for any  $f \in C_c(\mathbb{Z}; C_c(X))$  the expression  $(\omega \star \sum_{k=-N}^N q^{-2k} \omega_{k,k}^0)(f)$  converges as  $N \rightarrow \infty$ . By definition, the resulting expression is  $(\omega \star h)(f)$ . We choose  $C_c(\mathbb{Z}; C_c(X)) \subset C_0(X) \times_\tau \mathbb{Z}$  because  $h$  is defined on  $C_c(\mathbb{Z}; C_c(X))$ , see Remark 2.5. A similar definition is used for  $h \star \omega$ .

**Theorem 2.9.** *Let  $\xi \in L^2(X, \mu)$  be a finite linear combination of the basis elements  $f_{x,m}$ ,  $x, m \in \mathbb{Z}$ , see (2.6), and let  $\omega_\xi(f) = \langle \pi(f)\xi, \xi \rangle_{L^2(X, \mu)}$  be the corresponding element from  $B \subset (C_0(X) \times_\tau \mathbb{Z})^*$ . Then  $h$  is right and left invariant in the sense that  $(\omega_\xi \star h)(f) = \|\xi\|^2 h(f) = (h \star \omega_\xi)(f)$  for all  $f$  in the dense subspace  $C_c(\mathbb{Z}; C_c(X))$  of  $C_0(X) \times_\tau \mathbb{Z}$ .*

The theorem can be extended using the same argument to  $\omega_{\xi, \eta} \star h = \langle \xi, \eta \rangle h = h \star \omega_{\xi, \eta}$  for  $\omega_{\xi, \eta}(f) = \langle \pi(f)\xi, \eta \rangle_{L^2(X, \mu)}$ , where  $\xi, \eta$  are finite linear combinations of the basis elements  $f_{x,m}$ ,  $x, m \in \mathbb{Z}$ .

We start by proving a crucial special case.

**Lemma 2.10.**  *$h \star \omega_{r,s}^y = \delta_{y,0} \delta_{r,s} h$  and  $\omega_{k,l}^x \star h = \delta_{x,0} \delta_{k,l} h$  on  $C_c(\mathbb{Z}; C_c(X))$ .*

*Proof.* Take  $f \in C_c(\mathbb{Z}; C_c(X))$  and consider

$$\sum_{k=-N}^N q^{-2k} (\omega_{k,k}^0 \star \omega_{r,s}^y)(f) = \delta_{y,r-s} (-1)^y \sum_{n=-\infty}^{\infty} \left( \sum_{k=-N}^N q^{-2k} f_{n+s}^{s-k}(s) f_{n+r}^{r-k}(r) \right) \omega_{n+r, n+s}^y(f).$$

Note that the sum over  $n$  is finite. Lemma 2.8 implies  $\{q^{-k} f_{n+s}^{s-k}(s)\}_{k=-\infty}^{\infty} \in \ell^2(\mathbb{Z})$ , since  $q^{-k} f_{n+s}^{s-k}(s) = \mathcal{O}(q^{-k(1+|n|)})$  as  $k \rightarrow -\infty$  and  $q^{-k} f_{n+s}^{s-k}(s) = \mathcal{O}(q^{\frac{1}{2}k(k-1)} q^{k(1-2s-n)})$  as  $k \rightarrow \infty$ . So we can take the limit  $N \rightarrow \infty$ . Recall that

$$(2.13) \quad \sum_{k=-\infty}^{\infty} q^{-2k} f_m^{s-k}(s) f_{m+r-s}^{r-k}(r) = \delta_{r,s} q^{-2m},$$

which is [11, (4.3)] for the special case  $\alpha$  and  $c$  replaced by  $s - m$  and  $q^{2-2s}$  in base  $q^2$ . Now we can use (2.13) to find  $(h \star \omega_{r,s}^y)(f) = \delta_{y,r-s} \delta_{r,s} \sum_{n=-\infty}^{\infty} q^{-2n} \omega_{n,n}^0(f)$ , which is the desired result.

For the other statement we proceed analogously, now using

$$\sum_{r=-\infty}^{\infty} f_{n+l-k}^{r-l}(r) f_n^{r-k}(r) q^{-2r} = \delta_{k,l} q^{-2n},$$

which is the same sum as (2.13) using  $f_m^{-x}(k) = f_m^x(k+x)$ , see Lemma 2.8.  $\square$

*Proof of Theorem 2.9.* Let  $\xi = \sum_{x,s=-\infty}^{\infty} \xi_{x,s} f_{x,s} \in L^2(X, \mu)$  with only finitely many  $\xi_{x,s} \neq 0$ , then  $\omega_\xi = \sum_{r,s,y \in \mathbb{Z}} \left( \sum_{x-x'=y} \xi_{x',s} \bar{\xi}_{x,r} \right) \omega_{r,s}^y$ , so that for  $f \in C_c(\mathbb{Z}; C_c(X))$  we have

$$(2.14) \quad (h \star \omega_\xi)(f) = \sum_{r,s,x,n,k=-\infty}^{\infty} (-1)^{r-s} \bar{\xi}_{x,r} \xi_{x-r+s,s} q^{-2k} f_{n+s}^{s-k}(s) f_{n+r}^{r-k}(r) \omega_{n+r, n+s}^{r-s}(f),$$

provided that the sum is absolutely convergent. If this holds, we can use (2.13) to find the result. The Cauchy-Schwarz inequality applied to (2.13) gives

$$(2.15) \quad \sum_{k=-\infty}^{\infty} |q^{-2k} f_{n+s}^{s-k}(s) f_{n+r}^{r-k}(r)| \leq q^{-2n-r-s}.$$

Use of the estimate (2.15) for the sum over  $k$  leads to the termwise estimate

$$(2.16) \quad \sum_{r,s,x,n=-\infty}^{\infty} |\xi_{x,r}| |\xi_{x-r+s,s}| q^{-2n-r-s} |\omega_{n+r,n+s}^{r-s}(f)| = \\ \sum_{r,s=-\infty}^{\infty} |\omega_{r,s}^{r-s}(f)| q^{-s-r} \sum_{x,n=-\infty}^{\infty} |\xi_{x+r,r-n}| |\xi_{x+s,s-n}|,$$

for the right hand side of (2.14). The sum over  $x, n$  is estimated by  $\|\xi\|^2$ , and  $\omega_{r,s}^{r-s}(f) = 0$  for  $r > N$ ,  $|r-s| > M$  for some  $N, M \in \mathbb{N}$ . Hence, the sum is absolutely convergent for  $f \in C_c(\mathbb{Z}; C_c(X))$  and the result follows. For  $\omega_\xi \star h$  we proceed analogously.  $\square$

*Remark 2.11.* We can rewrite the Haar functional  $h$  in a coordinate free way. Use the co-variant representation  $\pi_{e^{i\theta}}$  of  $C_0(X) \times_\tau \mathbb{Z}$  to get  $\omega_{k,l}^x(f) = (2\pi)^{-1} \int_0^{2\pi} \langle \pi_{e^{i\theta}}(f) e_l, e_k \rangle e^{-ix\theta} d\theta$  and introduce the unbounded operator  $Q: \mathcal{D}(Q) \rightarrow \ell^2(\mathbb{Z})$ ,  $e_k \mapsto q^{-k} e_k$ , which is self-adjoint on its maximal domain  $\mathcal{D}(Q) = \{\sum_k c_k e_k \mid \sum_k |c_k|^2 q^{-2k} < \infty\}$ . We can rewrite  $h$  defined in Theorem 2.4 by

$$(2.17) \quad h(f) = \frac{1}{2\pi} \int_0^{2\pi} \text{Tr}_{|\ell^2(\mathbb{Z})} (\pi_{e^{i\theta}}(f) Q^2) d\theta$$

for any  $f \in C_0(X) \times_\tau \mathbb{Z}$  such that  $\pi_{e^{i\theta}}(f) Q^2$  is of trace class and  $\theta \mapsto \text{Tr}_{|\ell^2(\mathbb{Z})} (\pi_{e^{i\theta}}(f) Q^2)$  is integrable. Note that for any decomposable operator  $T \in \mathcal{B}(L^2(\mathbb{T}; \ell^2(\mathbb{Z})))$ , i.e.  $T = (2\pi)^{-1} \int_0^{2\pi} T(e^{i\theta}) d\theta$ , with  $T(e^{i\theta}) Q^2$  of trace class in  $\ell^2(\mathbb{Z})$  and  $\theta \mapsto \text{Tr}_{|\ell^2(\mathbb{Z})} (T(e^{i\theta}) Q^2)$  integrable we can define  $h(T)$  by (2.17). Note that the order of the operators in the trace in (2.17) is important. E.g. define the bounded operator  $S$  on  $\ell^2(\mathbb{Z})$  by  $S e_k = e_{-k}$  for  $k \geq 0$  and  $S e_k = 0$  for  $k < 0$ , then  $Q^2 S$  is of trace class and  $S Q^2$  is unbounded.

*Remark 2.12.* Recall that every element from the algebra  $A_q(SU(1,1))$  can be written uniquely as a sum of elements of the form  $\alpha^k \gamma^l p(\gamma^* \gamma)$ ,  $k, l \in \mathbb{Z}_{\geq 0}$ ,  $\alpha^k (\gamma^*)^l p(\gamma^* \gamma)$ ,  $k \in \mathbb{Z}_{\geq 0}$ ,  $l \in \mathbb{N}$ ,  $(\alpha^*)^k \gamma^l p(\gamma^* \gamma)$ ,  $l \in \mathbb{Z}_{\geq 0}$ ,  $k \in \mathbb{N}$ , and  $(\alpha^*)^k (\gamma^*)^l p(\gamma^* \gamma)$ ,  $k, l \in \mathbb{N}$ , where  $p$  is a polynomial, cf. Theorem 3.6. We can give a meaning to the Haar functional evaluated on such elements if we change  $p$  from polynomials to sufficiently decreasing functions. Let us do this explicitly for an element of the first type. Applying  $\pi_{e^{i\theta}}$  we see that  $\alpha^k \gamma^l p(\gamma^* \gamma)$  corresponds in the representation  $\pi_{e^{i\theta}}$  to the operator  $U^k e^{il\theta} (-Q^2; q^{-2})_k^{\frac{1}{2}} Q^l p(Q^2)$  on  $\ell^2(\mathbb{Z})$ , where  $U: \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$ ,  $e_k \mapsto e_{k+1}$ , is the unilateral shift. Hence, for a function  $p$  satisfying  $\sum_{r \in \mathbb{Z}} (-q^{-2r}; q^{-2})_k^{\frac{1}{2}} q^{-r(l+2)} |p(q^{-2r})| < \infty$  we see that the corresponding operator times  $Q^2$  is of trace class on  $\ell^2(\mathbb{Z})$ . Its trace is non-zero only for  $k = 0$ , and for  $k = 0$  we have  $\text{Tr}_{|\ell^2(\mathbb{Z})} e^{il\theta} Q^{l+2} p(Q^2) = e^{il\theta} \sum_{r \in \mathbb{Z}} q^{-r(l+2)} p(q^{-2r})$ . Integrating over the circle gives zero unless  $l = 0$ , hence for  $p$  sufficiently rapidly decreasing we have

$$h(\alpha^k \gamma^l p(\gamma^* \gamma)) = \delta_{k,0} \delta_{l,0} \sum_{r=-\infty}^{\infty} q^{-2r} p(q^{-2r}) = \delta_{k,0} \delta_{l,0} \frac{1}{1-q^2} \int_0^\infty p(x) d_{q^2} x,$$



where the last equality defines the Jackson  $q$ -integral on  $(0, \infty)$ . In a similar fashion the Haar functional applied to any of the other types of elements of  $A_q(SU(1,1))$  described in the beginning of this remark gives zero. So we see that Theorems 2.4 and 2.9 with Remark 2.11 correspond precisely to [41], [42, Lemme 2.2], [57]. So we have linked the Haar functional to the Jackson integral on  $(0, \infty)$  when restricted to the subalgebra corresponding to the self-adjoint element  $\gamma^*\gamma$ , see [21], [22], [57] for the further analysis.

### 3. THE QUANTISED UNIVERSAL ENVELOPING ALGEBRA AND SELF-ADJOINT ELEMENTS

In this section we gather the necessary algebraic results on the quantised universal enveloping algebra  $U_q(\mathfrak{su}(1,1))$ , which is the dual Hopf  $*$ -algebra to  $A_q(SU(1,1))$  introduced in §2, see [10] for generalities on quantised universal enveloping algebras and Hopf  $*$ -algebras. The proofs of all statements in this section are analogous to the corresponding statements for the compact quantum  $SU(2)$  group, see [25], [26], [27], [35], and are skipped or only indicated. The main idea is due to Koornwinder [35] resulting into a quantum group theoretic interpretation of a two-parameter family of the Askey-Wilson polynomials as spherical functions. Then Noumi and Mimachi, see [43], [45], [46], have given an interpretation of the full four parameter family of Askey-Wilson polynomials, see also [25], [26], [27]. As indicated by the results in [30] the algebraic methods apply to  $U_q(\mathfrak{su}(1,1))$  as well in case of the positive discrete series representations.

**3.1 The quantised universal enveloping algebra.** This subsection is a reminder and is used to fix the notation. The material of this subsection is standard, and we refer to e.g. [10] for further information. By  $U_q(\mathfrak{sl}(2, \mathbb{C}))$  we denote the algebra generated by  $A$ ,  $B$ ,  $C$  and  $D$  subject to the relations, where  $0 < q < 1$ ,

$$(3.1) \quad AD = 1 = DA, \quad AB = qBA, \quad AC = q^{-1}CA, \quad BC - CB = \frac{A^2 - D^2}{q - q^{-1}}.$$

It follows from (3.1) that the element

$$(3.2) \quad \Omega = \frac{q^{-1}A^2 + qD^2 - 2}{(q^{-1} - q)^2} + BC = \frac{q^{-1}D^2 + qA^2 - 2}{(q - q^{-1})^2} + CB$$

is a central element of  $U_q(\mathfrak{sl}(2, \mathbb{C}))$ , the Casimir element. The algebra  $U_q(\mathfrak{sl}(2, \mathbb{C}))$  is in fact a Hopf algebra with comultiplication  $\Delta: U_q(\mathfrak{sl}(2, \mathbb{C})) \rightarrow U_q(\mathfrak{sl}(2, \mathbb{C})) \otimes U_q(\mathfrak{sl}(2, \mathbb{C}))$  given by

$$\Delta(A) = A \otimes A, \quad \Delta(B) = A \otimes B + B \otimes D, \quad \Delta(C) = A \otimes C + C \otimes D, \quad \Delta(D) = D \otimes D.$$

The Hopf algebra  $U_q(\mathfrak{sl}(2, \mathbb{C}))$  is in duality with the Hopf algebra  $A_q(SL(2, \mathbb{C}))$  of the previous section, where the duality is incorporated by the representations of Theorem 3.1. There are two ways to introduce a  $*$ -operator in order to make  $U_q(\mathfrak{sl}(2, \mathbb{C}))$  a Hopf  $*$ -algebra. The first one is defined by its action on the generators as follows;  $A^* = A$ ,  $B^* = -C$ ,  $C^* = -B$ ,  $D^* = D$ . We call the corresponding Hopf  $*$ -algebra  $U_q(\mathfrak{su}(1,1))$ . The other  $*$ -structure is given by  $A^\times = A$ ,  $B^\times = C$ ,  $C^\times = B$ ,  $D^\times = D$ . The corresponding Hopf  $*$ -algebra is denoted by  $U_q(\mathfrak{su}(2))$ . Note that  $\Omega^* = \Omega = \Omega^\times$ .

**Theorem 3.1.** (See [10, Ch. 10]) *For each spin  $l \in \frac{1}{2}\mathbb{Z}_{\geq 0}$  there exists a unique  $(2l+1)$ -dimensional representation of  $U_q(\mathfrak{sl}(2, \mathbb{C}))$  such that the spectrum of  $A$  is contained in  $q^{\frac{1}{2}\mathbb{Z}}$ . Equip  $\mathbb{C}^{2l+1}$  with orthonormal basis  $\{e_n^l\}$ ,  $n = -l, -l+1, \dots, l$  and denote the representation by  $t^l$ . The action of the generators is given by*

$$(3.3) \quad \begin{aligned} t^l(A) e_n^l &= q^{-n} e_n^l, & t^l(D) e_n^l &= q^n e_n^l, \\ t^l(B) e_n^l &= \frac{\sqrt{(q^{-l+n-1} - q^{l-n+1})(q^{-l-n} - q^{l+n})}}{q^{-1} - q} e_{n-1}^l \\ t^l(C) e_n^l &= \frac{\sqrt{(q^{-l+n} - q^{l-n})(q^{-l-n-1} - q^{l+n+1})}}{q^{-1} - q} e_{n+1}^l, \end{aligned}$$

where  $e_{l+1}^l = 0 = e_{-l-1}^l$ .

The representation  $t^l$  of  $U_q(\mathfrak{sl}(2, \mathbb{C}))$  is not a  $*$ -representation of  $U_q(\mathfrak{su}(1, 1))$ , but it is a  $*$ -representation of  $U_q(\mathfrak{su}(2))$ .

Then  $A_q(SL(2, \mathbb{C}))$  is spanned by the matrix elements  $X \mapsto t_{n,m}^l(X) = \langle t^l(X) e_m^l, e_n^l \rangle$ , and the link is given by

$$(3.4) \quad t^{\frac{1}{2}} = \begin{pmatrix} t^{\frac{1}{2}}_{-\frac{1}{2}, -\frac{1}{2}} & t^{\frac{1}{2}}_{-\frac{1}{2}, \frac{1}{2}} \\ t^{\frac{1}{2}}_{\frac{1}{2}, -\frac{1}{2}} & t^{\frac{1}{2}}_{\frac{1}{2}, \frac{1}{2}} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Also

$$(3.5) \quad t^1 = \begin{pmatrix} \alpha^2 & \sqrt{1+q^2}\beta\alpha & \beta^2 \\ \sqrt{1+q^2}\gamma\alpha & 1 + (q+q^{-1})\beta\gamma & \sqrt{1+q^2}\delta\beta \\ \gamma^2 & \sqrt{1+q^2}\delta\gamma & \delta^2 \end{pmatrix}.$$

**3.2 Self-adjoint elements in  $A_q(SU(1, 1))$ .** The definition of the self-adjoint elements in  $A_q(SU(1, 1))$  we give here is strongly motivated by the paper by Koornwinder [35] for the compact quantum  $SU(2)$  group and the results of [30] for the positive discrete series representations of  $U_q(\mathfrak{su}(1, 1))$ . We define

$$(3.6) \quad Y_s = q^{\frac{1}{2}}B - q^{-\frac{1}{2}}C + \frac{s + s^{-1}}{q^{-1} - q}(A - D) \in U_q(\mathfrak{su}(1, 1)),$$

then  $Y_s$  is twisted primitive, i.e.  $\Delta(Y_s) = A \otimes Y_s + Y_s \otimes D$ , and  $Y_s A = (Y_s A)^*$  is self-adjoint for  $s \in \mathbb{R} \setminus \{0\}$ , and without loss of generality we assume  $|s| \geq 1$ . The convention is

$$(3.7) \quad Y_\infty = \lim_{s \rightarrow 0} s(q^{-1} - q) Y_s = \lim_{s \rightarrow \infty} s^{-1}(q^{-1} - q) Y_s = A - D.$$

The definition of  $Y_s$  is as in [30].

Then  $t^l(Y_s A) \in \text{Mat}_{2l+1}(\mathbb{C})$  is completely diagonalisable. Since the proof is completely analogous to the proof of [35, Thm. 4.3], we do not give the proof here.

**Lemma 3.2.** *The tridiagonal matrix  $t^l(Y_s A)$  is completely diagonalisable with spectrum*

$$\lambda_y(s) = \frac{sq^{2y} + s^{-1}q^{-2y} - (s + s^{-1})}{q^{-1} - q}, \quad y \in \{-l, -l+1, \dots, l\},$$

and corresponding eigenvector

$$v_y^l(s) = \sum_{n=-l}^l \frac{(q^{4l}; q^{-2})_{l-n}^{\frac{1}{2}}}{(q^2; q^2)_{l-n}^{\frac{1}{2}}} q^{\frac{1}{2}(l-n)(l-n-1)} s^{l-n} R_{l-n}(q^{2y-2l} + s^{-2}q^{-2y-2l}; s^{-2}, 2l; q^2) e_n^l$$

where

$$R_n(q^{-x} + cq^{x-N}; c, N; q) = {}_3\varphi_2 \left( \begin{matrix} q^{-n}, q^{-x}, cq^{x-N} \\ q^{-N}, 0 \end{matrix}; q, q \right)$$

is a dual  $q$ -Krawtchouk polynomial.

It is straightforward to check that  $t^l(Y_s A)^*$ , where  $*$  denotes the adjoint of a  $(2l+1) \times (2l+1)$ -matrix, equals  $-t^l(Y_{-s} A)$ , since  $(Y_s A)^\times = -Y_{-s} A$ . Note that  $\lambda_y(-s) = -\lambda_y(s)$ ,  $\lambda_y(s) = \lambda_{-y}(s^{-1})$ , and that  $\lambda_y(s) \neq \lambda_{y'}(s)$  for  $y \neq y'$ ,  $y, y' \in \{-l, \dots, l\}$  if  $s^2 \notin q^{2\mathbb{Z}}$ .

Next we define the matrix elements with respect to the basis of eigenvectors of  $t^l(Y_s A)$  by  $a_{i,j}^l(s, t)(X) = \langle t^l(X) v_j^l(t), v_i^l(-s) \rangle$  for real  $s, t$  satisfying  $|s|, |t| \geq 1$ , so that

$$a_{i,j}^l(s, t)(XY_t A) = \lambda_j(t) a_{i,j}^l(s, t)(X),$$

and

$$\begin{aligned} a_{i,j}^l(s, t)(Y_s A X) &= \langle t^l(X) v_j^l(t), (t^l(Y_s A))^* v_i^l(-s) \rangle \\ &= -\langle t^l(X) v_j^l(t), t^l(Y_{-s} A) v_i^l(-s) \rangle = -\lambda_i(-s) a_{i,j}^l(s, t)(X) = \lambda_i(s) a_{i,j}^l(s, t)(X). \end{aligned}$$

Or, using the notation  $X.\xi, \xi.X \in A_q(SL(2, \mathbb{C}))$  with  $X, Y \in U_q(\mathfrak{sl}(2, \mathbb{C}))$  for the elements defined by  $X.\xi(Y) = \xi(YX)$  and  $\xi.X(Y) = \xi(XY)$ , we have

$$(3.8) \quad \begin{aligned} (Y_t A).a_{i,j}^l(s, t) &= \lambda_j(t) a_{i,j}^l(s, t), & a_{i,j}^l(s, t).(Y_s A) &= \lambda_i(s) a_{i,j}^l(s, t), \\ Y_t.b_{i,j}^l(s, t) &= \lambda_j(t) D.b_{i,j}^l(s, t), & b_{i,j}^l(s, t).Y_s &= \lambda_i(s) b_{i,j}^l(s, t).D \end{aligned}$$

for  $b_{i,j}^l(s, t) = A.a_{i,j}^l(s, t)$ . Here we have used  $X.(Y.\xi) = (XY).\xi$ ,  $(\xi.X).Y = \xi.(XY)$  and  $(X.\xi).Y = X.(\xi.Y)$ , i.e. the applications  $X.\xi$  and  $\xi.X$  define mutual compatible left and right actions of  $U_q(\mathfrak{sl}(2, \mathbb{C}))$  on  $A_q(SL(2, \mathbb{C}))$ .

As before assume  $s, t \in \mathbb{R}$ ,  $|s|, |t| \geq 1$ . Write  $v_y^l(s) = \sum_{n=-l}^l v_y^l(s)_n e_n^l$ , then  $a_{i,j}^l(s, t) = \sum_{n,m=-l}^l v_j^l(t)_m \overline{v_i^l(-s)_n} t_{n,m}^l$  and  $b_{i,j}^l(s, t) = \sum_{n,m=-l}^l v_j^l(t)_m \overline{v_i^l(-s)_n} q^{-m} t_{n,m}^l$ , since we have  $A.t_{n,m}^l = q^{-m} t_{n,m}^l$ . The case  $l = \frac{1}{2}$  of Lemma 3.2 gives

$$v_{-\frac{1}{2}}^{\frac{1}{2}}(s) = s^{-1} e_{-\frac{1}{2}}^{\frac{1}{2}} + e_{\frac{1}{2}}^{\frac{1}{2}}, \quad v_{\frac{1}{2}}^{\frac{1}{2}}(s) = s e_{-\frac{1}{2}}^{\frac{1}{2}} + e_{\frac{1}{2}}^{\frac{1}{2}}$$

and using (3.4) we get

$$(3.9) \quad \begin{pmatrix} b_{-\frac{1}{2}, -\frac{1}{2}}^{\frac{1}{2}}(s, t) & b_{-\frac{1}{2}, \frac{1}{2}}^{\frac{1}{2}}(s, t) \\ b_{\frac{1}{2}, -\frac{1}{2}}^{\frac{1}{2}}(s, t) & b_{\frac{1}{2}, \frac{1}{2}}^{\frac{1}{2}}(s, t) \end{pmatrix} = \begin{pmatrix} -s^{-1}t^{-1}\alpha_{s,t} & -s^{-1}\beta_{s,t} \\ t^{-1}\gamma_{s,t} & \delta_{s,t} \end{pmatrix}$$

with

$$(3.10) \quad \begin{aligned} \alpha_{s,t} &= q^{\frac{1}{2}}\alpha + q^{-\frac{1}{2}}t\beta - q^{\frac{1}{2}}s\gamma - q^{-\frac{1}{2}}st\delta, & \beta_{s,t} &= q^{\frac{1}{2}}t\alpha + q^{-\frac{1}{2}}\beta - q^{\frac{1}{2}}st\gamma - q^{-\frac{1}{2}}s\delta, \\ \gamma_{s,t} &= -q^{\frac{1}{2}}s\alpha - q^{-\frac{1}{2}}st\beta + q^{\frac{1}{2}}\gamma + q^{-\frac{1}{2}}t\delta, & \delta_{s,t} &= -q^{\frac{1}{2}}st\alpha - q^{-\frac{1}{2}}s\beta + q^{\frac{1}{2}}t\gamma + q^{-\frac{1}{2}}\delta \end{aligned}$$

Similarly, using the vector spanning the kernel of  $t^1(Y_s A)$ ;

$$v_0^1(s) = q^{-1}e_{-1}^1 + \frac{s + s^{-1}}{\sqrt{1 + q^2}}e_0^1 + e_1^1$$

and (3.5) we see that  $b_{0,0}^1(s, t)$  equals, up to an affine transformation,

$$(3.11) \quad \begin{aligned} \rho_{s,t} &= \frac{1}{2}(\alpha^2 + \delta^2 + q\gamma^2 + q^{-1}\beta^2 + (t + t^{-1})(q\delta\gamma + \beta\alpha) \\ &\quad - (s + s^{-1})(q\gamma\alpha + \delta\beta) - (t + t^{-1})(s + s^{-1})\beta\gamma) \\ &= \frac{1}{2}(\alpha^2 + (\alpha^*)^2 + q(\gamma^2 + (\gamma^*)^2) + q(t + t^{-1})(\alpha^*\gamma + \gamma^*\alpha) \\ &\quad - q(s + s^{-1})(\gamma\alpha + \alpha^*\gamma^*) - q(t + t^{-1})(s + s^{-1})\gamma\gamma^*). \end{aligned}$$

Remark that (3.8) remains valid for  $\rho_{s,t}$  instead of  $b_{0,0}^1(s, t)$ , since  $1 \in A_q(SL(2, \mathbb{C}))$  satisfies  $Y_t.1 = 0 = 1.Y_s$ . Observe that  $\rho_{s,t} = \rho_{s\pm 1, t\pm 1} = \rho_{s,t}^*$ , since  $s$  and  $t$  are real.

*Remark 3.3.* There is also a certain symmetry between  $s$  and  $t$ . To be explicit, let  $A_q(SU(1, 1))^{\text{opp}}$  be the opposite Hopf  $*$ -algebra, see e.g. [10], then interchanging  $\gamma$  and  $\gamma^*$  gives a Hopf  $*$ -algebra isomorphism  $\psi: A_q(SU(1, 1)) \rightarrow A_q(SU(1, 1))^{\text{opp}}$  which maps  $\rho_{s,t}$  to  $\rho_{-t, -s}$ .

We calculate the Haar weight on the subalgebra generated by the self-adjoint element  $\rho_{s,t} \in A_q(SU(1, 1))$  in §5 explicitly.

The following limit case plays an important role in the sequel;

$$(3.12) \quad \begin{aligned} \rho_{\infty, t} &= \lim_{s \rightarrow 0} \frac{-2s}{q} \rho_{s,t} = \lim_{s \rightarrow \pm\infty} \frac{-2}{qs} \rho_{s,t} = \gamma\alpha + q^{-1}\delta\beta + (t + t^{-1})q^{-1}\beta\gamma \\ &= \alpha^*\gamma^* + \gamma\alpha + (t + t^{-1})\gamma\gamma^*. \end{aligned}$$

This element also satisfies  $\rho_{\infty, t} = \rho_{\infty, t\pm 1} = \rho_{\infty, t}^*$ , and we calculate the Haar weight on the subalgebra generated by the self-adjoint element  $\rho_{\infty, t} \in A_q(SU(1, 1))$  in §4 explicitly. We need the appropriate limit case of (3.10):

$$(3.13) \quad \begin{aligned} \alpha_{\infty, t} &= \lim_{s \rightarrow 0} \alpha_{s,t} = q^{\frac{1}{2}}\alpha + q^{-\frac{1}{2}}t\beta, & \beta_{\infty, t} &= \lim_{s \rightarrow 0} \beta_{s,t} = q^{\frac{1}{2}}t\alpha + q^{-\frac{1}{2}}\beta, \\ \gamma_{\infty, t} &= \lim_{s \rightarrow 0} \gamma_{s,t} = q^{\frac{1}{2}}\gamma + q^{-\frac{1}{2}}t\delta, & \delta_{\infty, t} &= \lim_{s \rightarrow 0} \delta_{s,t} = q^{\frac{1}{2}}t\gamma + q^{-\frac{1}{2}}\delta. \end{aligned}$$

Note that we can express the elements defined in (3.10) in terms of these elements by

$$(3.14) \quad \begin{aligned} \alpha_{s,t} &= \alpha_{\infty, t} - s\gamma_{\infty, t}, & \beta_{s,t} &= \beta_{\infty, t} - s\delta_{\infty, t}, \\ \gamma_{s,t} &= \gamma_{\infty, t} - s\alpha_{\infty, t}, & \delta_{s,t} &= \delta_{\infty, t} - s\beta_{\infty, t}. \end{aligned}$$

**3.3 Cartan decomposition.** The matrix elements  $t_{n,m}^l$ ,  $l \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ ,  $n, m \in \{-l, -l+1, \dots, l\}$  form a linear basis for  $A_q = A_q(SL(2, \mathbb{C}))$ . Put  $A_q^l = \text{span}_{\mathbb{C}}\{t_{n,m}^l \mid n, m = -l, \dots, l\}$  for  $l \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ . Note that  $b_{n,m}^l(s, t)$ ,  $n, m \in \{-l, -l+1, \dots, l\}$ , form a basis for  $A_q^l(SL(2, \mathbb{C}))$  as well if  $s^2, t^2 \notin q^{2\mathbb{Z}}$ .

**Proposition 3.4.** *Let  $s^2, t^2 \notin q^{2\mathbb{Z}}$ .*

(i) *Let  $\xi \in A_q^l$ ,  $l \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ , be a  $(s, t)$ -spherical element, i.e.  $Y_t \cdot \xi = 0 = \xi \cdot Y_s$ , and let  $\eta \in A_q$  satisfy*

$$(3.15) \quad Y_t \cdot \eta = \lambda D \cdot \eta \quad \text{and} \quad \eta \cdot Y_s = \mu \eta \cdot D$$

*for some  $\lambda, \mu \in \mathbb{C}$ . Then  $\eta\xi$  satisfies (3.15) for the same  $\lambda, \mu$ . Moreover, if  $\lambda, \mu \in \mathbb{R}$ , then  $\eta^* \eta$  is a  $(s, t)$ -spherical element.*

(ii) *If  $\eta \in A_q^l$  satisfies (3.15) for some  $\lambda, \mu \in \mathbb{C}$  and  $\eta$  is non-zero, then  $\lambda = \lambda_j(t)$ ,  $\mu = \lambda_i(s)$  for some  $i, j \in \{-l, -l+1, \dots, l\}$  and  $\eta$  is a multiple of  $b_{i,j}^l(s, t)$ .*

It follows that  $\rho_{s,t}$  generates the  $*$ -subalgebra of  $A_q(SL(2, \mathbb{C}))$  of  $(s, t)$ -spherical elements for  $s^2, t^2 \notin q^{2\mathbb{Z}}$ .

**Proposition 3.5.** *Let  $\eta \in A_q$  satisfy (3.15) with  $\lambda = \lambda_j(t)$  and  $\mu = \lambda_i(s)$ , then*

- (i)  $\alpha_{sq^{2i}, tq^{2j}} \eta$  satisfies (3.15) with  $\lambda = \lambda_{j-1/2}(t)$  and  $\mu = \lambda_{i-1/2}(s)$ ,
- (ii)  $\beta_{sq^{2i}, tq^{2j}} \eta$  satisfies (3.15) with  $\lambda = \lambda_{j+1/2}(t)$  and  $\mu = \lambda_{i-1/2}(s)$ ,
- (iii)  $\gamma_{sq^{2i}, tq^{2j}} \eta$  satisfies (3.15) with  $\lambda = \lambda_{j-1/2}(t)$  and  $\mu = \lambda_{i+1/2}(s)$ ,
- (iv)  $\delta_{sq^{2i}, tq^{2j}} \eta$  satisfies (3.15) with  $\lambda = \lambda_{j+1/2}(t)$  and  $\mu = \lambda_{i+1/2}(s)$ .

*In case  $s = \infty$  the result remains valid with  $Y_\infty$  defined in (3.7) and  $\lambda_j(\infty) = q^{2j} - 1$ .*

We skip the proofs of Propositions 3.4 and 3.5, since they are completely analogous to the proofs of [26, Prop. 6.4, Prop. 6.5], see also [25, Prop. 2.3]. We note that Proposition 3.5 can also be proved by direct calculations using (3.9),  $Y_t = Y_{tq^{2j}} - \lambda_j(t)(A - D)$ ,  $Y_t$  being twisted primitive and  $\lambda_i(s) + \lambda_{\pm \frac{1}{2}}(sq^{2i}) = \lambda_{i \pm \frac{1}{2}}(s)$ .

A direct consequence of Propositions 3.4 and 3.5 and Lemma 3.2 is the product structure of the matrix elements  $b_{i,j}^l(s, t)$  with  $\max(|i|, |j|) = l$ . For this we define elements  $\Gamma_{l,m}^{(i)}(s, t) \in A_q(SU(1, 1))$  for  $m \in \{-l, -l+1, \dots, l\}$ ,  $l \in \frac{1}{2}\mathbb{N}$  in terms of products of elementary elements with the convention  $\prod_{i=0}^k \xi_i = \xi_k \xi_{k-1} \dots \xi_0$  and the empty product being 1;

$$(3.16) \quad \begin{aligned} \Gamma_{l,m}^{(1)}(s, t) &= \prod_{i=0}^{l+m-1} \delta_{sq^{l-m+i}, tq^{m-l+i}} \prod_{j=0}^{l-m-1} \gamma_{sq^j, tq^{-j}} = C_1 b_{l,m}^l(s, t), \\ \Gamma_{l,m}^{(2)}(s, t) &= \prod_{i=0}^{l-m-1} \alpha_{sq^{l+m-i}, tq^{-l-m-i}} \prod_{j=0}^{l+m-1} \gamma_{sq^j, tq^{-j}} = C_2 b_{m,-l}^l(s, t), \\ \Gamma_{l,m}^{(3)}(s, t) &= \prod_{i=0}^{l+m-1} \delta_{sq^{m-l+i}, tq^{l-m+i}} \prod_{j=0}^{l-m-1} \beta_{sq^{-j}, tq^j} = C_3 b_{m,l}^l(s, t), \\ \Gamma_{l,m}^{(4)}(s, t) &= \prod_{i=0}^{l+m-1} \beta_{sq^{-l+m-i}, tq^{-l+m+i}} \prod_{j=0}^{l-m-1} \alpha_{sq^{-j}, tq^{-j}} = C_4 b_{-l,m}^l(s, t) \end{aligned}$$

for certain non-zero constants  $C_i$ . Initially, the second equality in each line of (3.16) holds for  $s^2, t^2 \notin q^{2\mathbb{Z}}$ , and this condition can be removed by continuity. An analogous expression as in (3.16) holds for the case  $s = \infty$ , where  $\Gamma_{l,m}^{(i)}(\infty, t) = \lim_{s \rightarrow \infty} \Gamma_{l,m}^{(i)}(s, t)$  by (3.13).

The explicit expression of  $b_{i,j}^l(s, t)$  for  $\max(|i|, |j|) = l$  in (3.16) and Propositions 3.4 and 3.5 imply the following Cartan-type decomposition of  $A_q(SL(2, \mathbb{C}))$ .

**Theorem 3.6.** *Let  $s^2, t^2 \notin q^{2\mathbb{Z}}$ .  $A_q(SU(1, 1))$  is a free right  $\mathbb{C}[\rho_{s,t}]$ -module, with  $\mathbb{C}[\rho_{s,t}]$ -basis given by*

$$\{1\} \cup \{\Gamma_{l,m}^{(1)}\}_{l \in \frac{1}{2}\mathbb{N}, m \in I_1^l} \cup \{\Gamma_{l,m}^{(2)}\}_{l \in \frac{1}{2}\mathbb{N}, m \in I_2^l} \cup \{\Gamma_{l,m}^{(3)}\}_{l \in \frac{1}{2}\mathbb{N}, m \in I_3^l} \cup \{\Gamma_{l,m}^{(4)}\}_{l \in \frac{1}{2}\mathbb{N}, m \in I_4^l},$$

where  $I_1^l = I_4^l = \{1-l, 2-l, \dots, l\}$ ,  $I_2^l = \{-l, 1-l, \dots, l\}$ , and  $I_3^l = \{1-l, 2-l, \dots, l-1\}$ .

So any element  $\xi \in A_q(SU(1, 1))$  can be written as a finite sum of the form

$$(3.17) \quad \xi = p(\rho_{s,t}) + \sum_{i=1}^4 \sum_{l \in \frac{1}{2}\mathbb{N}} \sum_{m \in I_i^l} \Gamma_{l,m}^{(i)}(s, t) p_{l,m}^{(i)}(\rho_{s,t})$$

for uniquely determined polynomials  $p, p_{l,m}^{(i)}$ .

*Remark 3.7.* (i) We also have a corresponding decomposition for the case  $s = \infty$ , and for the case  $(s, t) = (\infty, \infty)$  we are back to the case discussed in [41], [42], see also Remark 2.12. Note that  $h(\xi)$  with  $\xi$  as in (3.17) is not well-defined, but it can be defined properly after replacing the polynomials in (3.17) by sufficiently decreasing functions, cf. Remark 2.12. For  $(s, t) = (\infty, \infty)$  the Cartan decomposition is formally orthogonal with respect to the Haar functional  $h$  by  $\langle \xi_1, \xi_2 \rangle = h(\xi_2^* \xi_1)$ , see [21]. For the general case  $(s, t)$  this is not clear.

(ii) The Cartan decomposition of Theorem 3.6 is the decomposition of  $A_q(SU(1, 1))$  into common eigenspaces of the left action of  $AY_t$  and right action of  $Y_s A$  on  $A_q(SU(1, 1))$ , i.e. of the left and right infinitesimal action of a “torus” depending on a parameter. Since the Casimir operator  $\Omega$  defined in (3.2) commutes with these actions, the Casimir operator preserves the Cartan decomposition.

(iii) For  $\xi$  as in (3.17) we have

$$(3.18) \quad \pi(\xi) = p(\pi(\rho_{s,t})) + \sum_{i,l,m} \pi(\Gamma_{l,m}^{(i)}(s, t)) p_{l,m}^{(i)}(\pi(\rho_{s,t}))$$

as an unbounded operator on  $L^2(X, \mu)$ . Now  $\pi(\rho_{s,t})$  is a symmetric unbounded operator. Suppose that  $\mathcal{D}(s, t)$  is the domain of a self-adjoint extension of  $\pi(\rho_{s,t})$  which is preserved by  $\pi(\Gamma_{l,m}^{(i)}(s, t))$ , then the right hand side of (3.18) makes sense as an unbounded linear operator on  $L^2(X, \mu)$  with domain  $\mathcal{D}(s, t)$  for all continuous functions  $p, p_{l,m}^{(i)}$  by the functional calculus of unbounded operators. If the functions  $p, p_{l,m}^{(i)}$  are such that the right hand side of (3.18) are in  $\pi(N_h)$ , the formal decomposition (3.18) of the corresponding unique element  $\xi \in N_h$  is called the Cartan decomposition of  $\xi$ .

It follows from (3.16), (3.8), Proposition 3.4 and since  $\rho_{s,t}$  generates the algebra of  $(s, t)$ -spherical elements that  $(\Gamma_{l,m}^{(i)}(s, t))^* \Gamma_{l,m}^{(i)}(s, t)$  is a polynomial in  $\rho_{s,t}$ . From (3.16) and (3.11) we see that the degree of this polynomial is  $2l$ . If we use the one-dimensional  $*$ -representation of  $A_q(SU(1, 1))$  sending  $\alpha$  to  $e^{\frac{1}{2}i\theta}$  and  $\gamma$  to zero,  $\theta \in \mathbb{R}$ , we obtain

$$(3.19) \quad \begin{aligned} \Gamma_{l,m}^{(1)}(s, t)^* \Gamma_{l,m}^{(1)}(s, t) &= C_1(qste^{i\theta}, qste^{-i\theta}; q^2)_{l+m} (q\frac{s}{t}e^{i\theta}, q\frac{s}{t}e^{-i\theta}; q^2)_{l-m} |_{\cos \theta = \rho_{s,t}}, \\ \Gamma_{l,m}^{(2)}(s, t)^* \Gamma_{l,m}^{(2)}(s, t) &= C_2 s^{2l-2m} (q\frac{s}{t}e^{i\theta}, q\frac{s}{t}e^{-i\theta}; q^2)_{l+m} (\frac{q}{st}e^{i\theta}, \frac{q}{st}e^{-i\theta}; q^2)_{l-m} |_{\cos \theta = \rho_{s,t}}, \\ \Gamma_{l,m}^{(3)}(s, t)^* \Gamma_{l,m}^{(3)}(s, t) &= C_3 s^{2l-2m} (qste^{i\theta}, qste^{-i\theta}; q^2)_{l+m} (q\frac{t}{s}e^{i\theta}, q\frac{t}{s}e^{-i\theta}; q^2)_{l-m} |_{\cos \theta = \rho_{s,t}}, \\ \Gamma_{l,m}^{(4)}(s, t)^* \Gamma_{l,m}^{(4)}(s, t) &= C_4 s^{4l} (q\frac{t}{s}e^{i\theta}, q\frac{t}{s}e^{-i\theta}; q^2)_{l+m} (\frac{q}{st}e^{i\theta}, \frac{q}{st}e^{-i\theta}; q^2)_{l-m} |_{\cos \theta = \rho_{s,t}}, \end{aligned}$$

for positive constants  $C_i$  independent of  $\theta$  and  $s$ , cf. [26, §7]. In (3.19) we also have the appropriate case for  $s = \infty$  using (3.12) and (3.13);

$$(3.20) \quad \begin{aligned} \Gamma_{l,m}^{(1)}(\infty, t)^* \Gamma_{l,m}^{(1)}(\infty, t) &= C_1(-tq^2\rho_{\infty,t}; q^2)_{l+m} (-q^2\rho_{\infty,t}/t; q^2)_{l-m}, \\ \Gamma_{l,m}^{(2)}(\infty, t)^* \Gamma_{l,m}^{(2)}(\infty, t) &= C_2(-q^2\rho_{\infty,t}/t; q^2)_{l+m} (-tq^{2+2m-2l}\rho_{\infty,t}; q^2)_{l-m}, \\ \Gamma_{l,m}^{(3)}(\infty, t)^* \Gamma_{l,m}^{(3)}(\infty, t) &= C_3(-q^2t\rho_{\infty,t}; q^2)_{l+m} (-q^{2-2l+2m}\rho_{\infty,t}/t; q^2)_{l-m}, \\ \Gamma_{l,m}^{(4)}(\infty, t)^* \Gamma_{l,m}^{(4)}(\infty, t) &= C_4(-q^{2-2l-2m}\rho_{\infty,t}/t; q^2)_{l+m} (-tq^{2-2l+2m}\rho_{\infty,t}; q^2)_{l-m}, \end{aligned}$$

for positive constants  $C_i$ .

**3.4 Factorisation and commutation results.** In order to obtain recurrence relations in later sections we need factorisation and commutation relations in the algebra  $A_q(SU(1, 1))$ . The following corollary is consequence of Proposition 3.4 and Proposition 3.5, see [27, §2] for the analogous statement for the case  $A_q(SU(2))$ . Since the proof is the same we skip it.

**Corollary 3.8.** *The following factorisation and commutation relations hold;*

$$\begin{aligned} \beta_{sq,tq^{-1}}\gamma_{s,t} &= -2st\rho_{s,t} + q^{-1}t^2 + qs^2, & \gamma_{sq^{-1},tq}\beta_{s,t} &= -2st\rho_{s,t} + qt^2 + q^{-1}s^2, \\ \alpha_{sq,tq}\delta_{s,t} &= -2qst\rho_{s,t} + 1 + q^2s^2t^2, & \delta_{sq^{-1},tq^{-1}}\alpha_{s,t} &= -2q^{-1}st\rho_{s,t} + 1 + q^{-2}s^2t^2, \end{aligned}$$

and

$$\begin{aligned} \alpha_{s,t}\rho_{s,t} &= \rho_{sq^{-1},tq^{-1}}\alpha_{s,t}, & \beta_{s,t}\rho_{s,t} &= \rho_{sq^{-1},tq}\beta_{s,t}, \\ \gamma_{s,t}\rho_{s,t} &= \rho_{sq,tq^{-1}}\gamma_{s,t}, & \delta_{s,t}\rho_{s,t} &= \rho_{sq,tq}\delta_{s,t}. \end{aligned}$$

Combining Corollary 3.8 with (3.14) gives

$$(3.21) \quad -2st\rho_{s,t} + q^{-1}t^2 + qs^2 = \beta_{sq,tq^{-1}}\gamma_{s,t} = (\beta_{\infty,tq^{-1}} - sq\delta_{\infty,tq^{-1}})(\gamma_{\infty,t} - s\alpha_{\infty,t}),$$

which is one of many ways of writing  $\rho_{s,t}$  in products of matrix elements  $b_{i,j}^{\frac{1}{2}}(\infty, t)$ .

The limit case  $s = \infty$ , i.e.  $s \rightarrow 0$ , of Corollary 3.8 immediately gives the following.

**Corollary 3.9.** *The following factorisation and commutation relations hold;*

$$\begin{aligned} q^{-1}\beta_{\infty,tq^{-1}}\gamma_{\infty,t} &= t\rho_{\infty,t} + q^{-2}t^2, & q^{-1}\gamma_{\infty,tq}\beta_{\infty,t} &= t\rho_{\infty,t} + t^2, \\ q^{-2}\alpha_{\infty,tq}\delta_{\infty,t} &= t\rho_{\infty,t} + q^{-2}, & \delta_{\infty,tq^{-1}}\alpha_{\infty,t} &= t\rho_{\infty,t} + 1, \end{aligned}$$

and

$$\begin{aligned} \alpha_{\infty,t}\rho_{\infty,t} &= q\rho_{\infty,tq^{-1}}\alpha_{\infty,t}, & \beta_{\infty,t}\rho_{\infty,t} &= q\rho_{\infty,tq}\beta_{\infty,t}, \\ \gamma_{\infty,t}\rho_{\infty,t} &= q^{-1}\rho_{\infty,tq^{-1}}\gamma_{\infty,t}, & \delta_{\infty,t}\rho_{\infty,t} &= q^{-1}\rho_{\infty,tq}\delta_{\infty,t}. \end{aligned}$$

#### 4. THE HAAR FUNCTIONAL ON THE ALGEBRA GENERATED BY $\rho_{\infty,t}$

The Haar functional on the Cartan decomposition of Theorem 3.6 for  $(s, t) = (\infty, \infty)$  is related to the Jackson integral on  $(0, \infty)$ , see [21], [22], [42], [57] and Remark 2.12. In this section we show that the Haar functional on the Cartan decomposition of Theorem 3.6 for the case  $s = \infty$ ,  $t > q^{-1}$  finite, is related to the Jackson integral on  $[-d, \infty)$  for some  $d > 0$ . The key ingredient is the spectral analysis of the unbounded symmetric operator  $\pi_{e^{i\theta}}(\rho_{\infty,t})$  given in [11] yielding an orthogonal basis of eigenvectors of  $\ell^2(\mathbb{Z})$ . To use the expression for  $h$  of Remark 2.11 we have to calculate the matrix elements of  $Q^2$  in this basis of eigenvectors in order to calculate the trace. We also show that the elements of (3.13) in the representation  $\pi_{e^{i\theta}}$  act as shift operators in the basis of eigenvectors. The results and approach are motivated by the results for the quantum  $SU(2)$  group case considered in [31, §5] and we consider the Jackson integral on  $[-d, \infty)$  as the non-compact analogue of the Jackson integral on  $[-d, c]$ . The proofs are more involved due to the unboundedness of the operators.

**4.1 Spectral analysis of  $\pi_{e^{i\theta}}(\rho_{\infty,t})$ .** Using Proposition 2.1 and (3.12) we get

$$\pi_{e^{i\theta}}(\rho_{\infty,t})e_k = (t + t^{-1})q^{-2k}e_k + e^{i\theta}q^{-1-k}\sqrt{1 + q^{-2k}}e_{k+1} + e^{-i\theta}q^{-k}\sqrt{1 + q^{2-2k}}e_{k-1},$$

and by going over to the orthonormal basis  $f_k = e^{ik\theta}e_k$  we obtain

$$\pi_{e^{i\theta}}(\rho_{\infty,t})f_k = (t + t^{-1})q^{-2k}f_k + q^{-1-k}\sqrt{1 + q^{-2k}}f_{k+1} + q^{-k}\sqrt{1 + q^{2-2k}}f_{k-1}.$$

This operator is unbounded and symmetric on the domain consisting of finite linear combinations of the basis vectors. This unbounded symmetric operator has been studied in detail in [11] for  $t \in \mathbb{R} \setminus \{0\}$ . It turns out that the operator in question is essentially self-adjoint for  $|t| \geq q^{-1}$ . Then the spectrum consists completely of point spectrum plus one accumulation point at zero, which itself is not in the point spectrum. The domain of  $\pi_{e^{i\theta}}(\rho_{\infty,t})$  is its maximal domain, i.e.  $\{\sum_k c_k e_k \in \ell^2(\mathbb{Z}) \mid \sum_k c_k \pi_{e^{i\theta}}(\rho_{\infty,t})e_k \in \ell^2(\mathbb{Z})\}$ . Proposition 4.1 is the analogue of [31, Prop. 5.2] and has been proved in [11].



**Proposition 4.1.** *Let  $t \in \mathbb{R}$  satisfy  $|t| > q$ . There exists an orthogonal basis of  $\ell^2(\mathbb{Z})$  of the form  $\{v_p^\theta(t) \mid p \in \mathbb{Z}_{\geq 0}\} \cup \{w_p^\theta(t) \mid p \in \mathbb{Z}\}$  given by*

$$v_p^\theta(t) = \sum_{k=-\infty}^{\infty} e^{ik\theta} V_k(-q^{2p}t^{-1}; t) e_k, \quad w_p^\theta(t) = \sum_{k=-\infty}^{\infty} e^{ik\theta} V_k(q^{2p}t; t) e_k,$$

where  $V_k(x; t) = (-q^{2-2k}; q^2)_{\infty}^{\frac{1}{2}} q^{\frac{1}{2}k(k+1)} (-t)^{-k} {}_1\varphi_1(-(xt)^{-1}; q^2t^{-2}; q^2, xq^{2k+2}t^{-1})$ . These vectors are all contained in the maximal domain of  $\pi_{e^{i\theta}}(\rho_{\infty,t})$ . Moreover,  $\pi_{e^{i\theta}}(\rho_{\infty,t}) v_p^\theta(t) = -q^{2p}t^{-1} v_p^\theta(t)$ ,  $p \in \mathbb{Z}_{\geq 0}$ , and  $\pi_{e^{i\theta}}(\rho_{\infty,t}) w_p^\theta(t) = q^{2p}t w_p^\theta(t)$ ,  $p \in \mathbb{Z}$ . The lengths of the orthogonal basis vectors are given by

$$\begin{aligned} \|v_p^\theta(t)\|^2 &= t^2 q^{-2p} \frac{(q^2; q^2)_p}{(q^2t^{-2}; q^2)_p} \frac{(q^2, -t^{-2}, -q^2t^2; q^2)_{\infty}}{(q^2t^{-2}; q^2)_{\infty}}, \quad p \in \mathbb{Z}_{\geq 0}, \\ \|w_p^\theta(t)\|^2 &= q^{-2p} \frac{(-q^{2+2p}, q^2, q^2, -t^{-2}, -q^2t^2; q^2)_{\infty}}{(-q^{2p+2}t^2, q^2t^{-2}, q^2t^{-2}; q^2)_{\infty}}, \quad p \in \mathbb{Z}. \end{aligned}$$

For  $|t| \geq q^{-1}$ , the operator  $\pi_{e^{i\theta}}(\rho_{\infty,t})$  is self-adjoint on its maximal domain.

In particular we find that the spectrum of  $\pi_{e^{i\theta}}(\rho_{\infty,t})$  consists of  $\sigma(\pi_{e^{i\theta}}(\rho_{\infty,t})) = \{-q^{2p}t^{-1}\}_{p \in \mathbb{Z}_{\geq 0}} \cup \{q^{2p}t\}_{p \in \mathbb{Z}} \cup \{0\}$ , which is independent of  $\theta$ . This also follows from  $T(e^{i\theta})\pi_{e^{i\theta}}(\rho_{\infty,t})T(e^{i\theta})^* = \pi_1(\rho_{\infty,t})$  with  $T(e^{i\theta})$  the unitary operator on  $\ell^2(\mathbb{Z})$  defined by  $e_k \mapsto e^{-ik\theta} e_k$ . Using [15, Ch. II.2, §6] we find for a bounded continuous function  $g$

$$(4.1) \quad g(\pi(\rho_{\infty,t})) = \frac{1}{2\pi} \int_0^{2\pi} g(\pi_{e^{i\theta}}(\rho_{\infty,t})) d\theta = T^*(\text{id} \otimes g(\pi_1(\rho_{\infty,t})))T,$$

$T = \frac{1}{2\pi} \int_0^{2\pi} T(e^{i\theta}) d\theta$ , using  $L^2(\mathbb{T}; \ell^2(\mathbb{Z})) \cong L^2(\mathbb{T}) \otimes \ell^2(\mathbb{Z})$  as tensor product of Hilbert spaces. So  $g(\pi(\rho_{\infty,t}))$  is a decomposable operator on  $\frac{1}{2\pi} \int_0^{2\pi} \ell^2(\mathbb{Z}) d\theta = L^2(\mathbb{T}; \ell^2(\mathbb{Z}))$ , since  $T$  commutes with multiplication by a function from  $L^2(\mathbb{T})$ , see [15, Ch. II.2, §5].

Note that Proposition 4.1 gives an orthogonal decomposition  $\ell^2(\mathbb{Z}) = V^\theta(t) \oplus W^\theta(t)$ , with  $V^\theta(t)$ , respectively  $W^\theta(t)$ , the closure of the linear span of the vectors  $v_p^\theta(t)$ ,  $p \in \mathbb{Z}_{\geq 0}$ , respectively  $w_p^\theta(t)$ ,  $p \in \mathbb{Z}$ . Using [15, Ch. II.1] we obtain the decomposition  $L^2(\mathbb{T}; \ell^2(\mathbb{Z})) = V(t) \oplus W(t)$  with  $V(t) = (2\pi)^{-1} \int_0^{2\pi} V^\theta(t) d\theta$  and  $W(t) = (2\pi)^{-1} \int_0^{2\pi} W^\theta(t) d\theta$ .

**4.2 Calculating the trace.** In this subsection we calculate the trace of  $g(\pi_{e^{i\theta}}(\rho_{\infty,t}))Q^2$ , cf. (2.17), for sufficiently decreasing function  $g$  using the basis described in Proposition 4.1. We start with the partial analogue of [31, Lemma 5.5]. The operator  $Q^2$  is self-adjoint with respect to its maximal domain  $\mathcal{D}(Q^2) = \{\sum_k c_k e_k \in \ell^2(\mathbb{Z}) \mid \sum_k |c_k|^2 q^{-4k} < \infty\}$ . The proof of Lemma 4.2 is a lengthy calculation, and can be skipped at first reading.

**Lemma 4.2.** *Let  $|t| > q^{-1}$ , then  $v_p^\theta(t), w_p^\theta(t) \in \mathcal{D}(Q^2)$ . Moreover,*

$$\frac{\langle Q^2 v_p^\theta(t), v_p^\theta(t) \rangle}{\langle v_p^\theta(t), v_p^\theta(t) \rangle} = \frac{q^{2p}}{t^2 - 1}, \quad p \in \mathbb{Z}_{\geq 0}, \quad \frac{\langle Q^2 w_p^\theta(t), w_p^\theta(t) \rangle}{\langle w_p^\theta(t), w_p^\theta(t) \rangle} = \frac{q^{2p}}{1 - t^{-2}}, \quad p \in \mathbb{Z}.$$

*Proof.* Since  $(-q^{2-2k}; q^2)_\infty q^{\frac{1}{2}k(k+1)} = \mathcal{O}(q^k)$  as  $k \rightarrow \infty$ , we see that  $V_k(x; t) = \mathcal{O}((q/t)^k)$  as  $k \rightarrow \infty$ . In order to have  $v_p^\theta(t), w_p^\theta(t) \in \mathcal{D}(Q^2)$  we need  $|qt|^{-k}$  to be square summable for  $k \rightarrow \infty$ . Hence, we need  $|t| > q^{-1}$ . Assuming that  $|t| > q^{-1}$  we see that  $Q^2 v_p^\theta(t)$  and  $Q^2 w_p^\theta(t)$  are well-defined elements of  $\ell^2(\mathbb{Z})$ .

The calculation of the diagonal elements of  $Q^2$  in this basis is based on orthogonality properties that follow from Proposition 4.1. The idea of the proof is taken from the proof of [31, Lemma 5.5], but since it is much more involved we give the details. Let us consider the first case, and for this we introduce a moment functional for the  $q$ -Laguerre polynomials defined by

$$\mathcal{L}(f) = \sum_{k=-\infty}^{\infty} (-q^{2-2k}; q^2)_\infty q^{k(k+1)} t^{-2k} f(q^{2k}),$$

cf. [11, §4]. Let  $P_p(x) = {}_1\varphi_1(q^{-2p}; qt^{-2}; q^2, -xq^{2+2p}t^{-2})$  be the corresponding orthogonal polynomials, which are  $q$ -Laguerre polynomials in which the usual parameter  $\alpha$  of the  $q$ -Laguerre polynomials corresponds to  $t$  via  $t^{-2} = q^{2\alpha}$ . Then we have

$$\mathcal{L}(P_p P_m) = \delta_{mp} t^2 q^{-2p} \frac{(q^2; q^2)_p}{(q^2 t^{-2}; q^2)_p} \frac{(q^2, -t^{-2}, -q^2 t^2; q^2)_\infty}{(q^2 t^{-2}; q^2)_\infty}, \quad m, p \in \mathbb{Z}_{\geq 0},$$

which follows from Proposition 4.1, cf. [11]. It turns out that we can calculate the general matrix element  $\langle Q^2 v_p^\theta(t), v_m^\theta(t) \rangle$  without any extra difficulty. Since  $Q^2$  is self-adjoint and the vectors  $v_p^\theta(t)$  are in its domain, we can assume without loss of generality that  $m \leq p$ . The matrix element can be expressed in terms of the moment functional;

$$\langle Q^2 v_p^\theta(t), v_m^\theta(t) \rangle = \mathcal{L}(x^{-1} P_m(x) P_p(x)) = \text{CT}(P_m) \mathcal{L}(x^{-1} P_p(x)),$$

where  $\text{CT}(P_m)$  means the constant term of the polynomial  $P_m$ . This is valid since  $x^{-1} P_m(x) = \text{CT}(P_m) x^{-1} + \text{polynomial of degree less than } m$ , since we have  $\mathcal{L}(P' P_p) = 0$  for any polynomial  $P'$  of degree less than  $p$ . It remains to calculate

$$\begin{aligned} \mathcal{L}(x^{-1} P_p(x)) &= \sum_{k=-\infty}^{\infty} (-q^{2-2k}; q^2)_\infty q^{k(k-1)} t^{-2k} {}_1\varphi_1(q^{-2p}; q^2 t^{-2}; q^2, -q^{2k+2p+2} t^{-2}) \\ &= \sum_{r=0}^p \frac{(q^{-2p}; q^2)_r}{(q^2, q^2 t^{-2}; q^2)_r} q^{r(r-1)} t^{-2r} q^{2r(p+1)} \sum_{k=-\infty}^{\infty} (-q^{2-2k}; q^2)_\infty q^{k(k-1)} t^{-2k} q^{2rk}, \end{aligned}$$

where interchanging summations is allowed since the sum converges absolutely. Using the theta product identity (2.9) and Ramanujan's  ${}_1\psi_1$ -summation formula, see [17, (5.2.1)], we can evaluate the inner sum as

$$(t^{-2}; q^2)_r t^{2r} q^{-r(r-1)} \frac{(q^2, -t^{-2}, -q^2 t^2; q^2)_\infty}{(t^{-2}; q^2)_\infty}.$$

Since  $\text{CT}(P_m) = 1$  we obtain for  $m \leq p$

$$\begin{aligned} \langle Q^2 v_p^\theta(t), v_m^\theta(t) \rangle &= \frac{(q^2, -t^{-2}, -q^2 t^2; q^2)_\infty}{(t^{-2}; q^2)_\infty} {}_2\varphi_1(q^{-2p}, t^{-2}; q^2 t^{-2}; q^2, q^{2p+2}) \\ &= \frac{(q^2, -t^{-2}, -q^2 t^2; q^2)_\infty}{(t^{-2}; q^2)_\infty} \frac{(q^2; q^2)_p}{(q^2 t^{-2}; q^2)_p} \end{aligned}$$

by the  $q$ -Chu-Vandermonde summation, see [17, (1.5.2)]. Using the norms given in Proposition 4.1 we obtain

$$(4.2) \quad \frac{\langle Q^2 v_p^\theta(t), v_m^\theta(t) \rangle}{\|v_p^\theta(t)\| \|v_m^\theta(t)\|} = \frac{t^{-2} q^{p+m}}{1-t^{-2}} \left( \frac{(q^2; q^2)_p (q^2 t^{-2}; q^2)_m}{(q^2; q^2)_m (q^2 t^{-2}; q^2)_p} \right)^{\frac{1}{2}}.$$

Now take  $m = p$  to find the first statement.

For the other statement we proceed in the same way. However, this time we cannot get rid of a summation so easily since there are no orthogonal polynomials around. We consider the functions

$$\begin{aligned} M_p(x) &= {}_1\varphi_1(-q^{-2p}t^{-2}; q^2 t^{-2}; q^2, x q^{2p+2}) \\ &= \frac{(x q^{2p+2}; q^2)_\infty}{(q^2 t^{-2}; q^2)_\infty} {}_1\varphi_1(-x; x q^{2p+2}; q^2, q^2 t^{-2}) \end{aligned}$$

by the transformation (2.11). Using (2.9) we find

$$\begin{aligned} (4.3) \quad \langle Q^2 w_p^\theta(t), w_m^\theta(t) \rangle &= \mathcal{L}(x^{-1} M_p(x) M_m(x)) \\ &= \frac{(-1, -q^2; q^2)_\infty}{(q^2 t^{-2}; q^2)_\infty} \sum_{k=-\infty}^{\infty} \frac{t^{-2k}}{(-q^{2k}; q^2)_\infty} (q^{2k+2p+2}; q^2)_\infty {}_1\varphi_1 \left( \begin{matrix} -q^{2k} \\ q^{2k+2p+2} \end{matrix}; q^2, \frac{q^2}{t^2} \right) \\ &\quad \times (q^{2k+2m+2}; q^2)_\infty {}_1\varphi_1 \left( \begin{matrix} -q^{2k} \\ q^{2k+2m+2} \end{matrix}; q^2, \frac{q^2}{t^2} \right) \end{aligned}$$

Now consider the following generating functions, see [11, Lemma 5.1],

$$\sum_{k=-\infty}^{\infty} z^k b^k \frac{(q^{k+1}; q)_\infty}{(a q^k/b; q)_\infty} {}_1\varphi_1 \left( \begin{matrix} a q^k/b \\ q^{k+1} \end{matrix}; q, b x \right) = \frac{(q, a z, x/z; q)_\infty}{(a/b, b z; q)_\infty}, \quad 0 < |z| < |b|^{-1}$$

and

$$\sum_{n=-\infty}^{\infty} w^n (q^{n+1}; q)_\infty {}_1\varphi_1 \left( \begin{matrix} d q^{n+1}/y \\ q^{n+1} \end{matrix}; q, y \right) = \frac{(d, q, y/w; q)_\infty}{(w, d/w; q)_\infty}, \quad |d| < |w| < 1,$$

to see that for  $0 < |z| < |t|^2$  we have

$$\frac{(q^2, q^2/z, -t^{-2} q^{-2p} z; q^2)_\infty}{(-q^{-2p}, z t^{-2}; q^2)_\infty} z^{-p} t^{2p} = \sum_{k=-\infty}^{\infty} z^k t^{-2k} \frac{(q^{2k+2p+2}; q^2)_\infty}{(-q^{2k}; q^2)_\infty} {}_1\varphi_1 \left( \begin{matrix} -q^{2k} \\ q^{2k+2p+2} \end{matrix}; q^2, \frac{q^2}{t^2} \right),$$

and for  $1 < |z| < |t|^2 q^{2m}$  we have

$$\frac{(-t^{-2} q^{-2m}, q^2, z q^2 t^{-2}; q^2)_\infty}{(1/z, -t^{-2} q^{-2m} z; q^2)_\infty} z^m = \sum_{n=-\infty}^{\infty} z^{-n} (q^{2n+2m+2}; q^2)_\infty {}_1\varphi_1 \left( \begin{matrix} -q^{2n} \\ q^{2n+2m+2} \end{matrix}; q^2, \frac{q^2}{t^2} \right).$$

Note  $|t| > q^{-1}$  by assumption and assume first the additional condition  $|t|^2 q^{2m} > 1$ , so that multiplying these generating functions is valid in the annulus  $1 < |z| < \min(|t|^2, |t|^2 q^{2m})$ . The constant term is the series in (4.3). Hence, this series equals the constant term of

$$\frac{(q^2, q^2, -t^{-2}q^{-2m}; q^2)_\infty t^{2p}}{(-q^{-2p}; q^2)_\infty} \frac{z^{m-p}(-t^{-2}q^{-2p}z; q^2)_{p-m}}{(1-1/z)(1-zt^{-2})},$$

assuming without loss of generality that  $p \geq m$ . Since  $1 < |z| < t^2$  we have

$$\frac{1}{(1-1/z)(1-zt^{-2})} = \sum_{k=0}^{\infty} z^{-k} \sum_{p=0}^{\infty} z^p t^{-2p} = \frac{1}{1-t^{-2}} \left( \sum_{l=-\infty}^{-1} z^l + \sum_{l=0}^{\infty} t^{-2l} z^l \right)$$

and by the  $q$ -binomial formula [17, (1.3.14)] we have

$$z^{m-p}(-t^{-2}q^{-2p}z; q^2)_{p-m} = \sum_{k=0}^{p-m} \frac{(q^{2m-2p}; q^2)_k}{(q^2; q^2)_k} z^{m-p+k} (-t^{-2}q^{-2m})^k.$$

So

$$\begin{aligned} \text{CT} \left( \frac{z^{m-p}(-t^{-2}q^{-2p}z; q^2)_{p-m}}{(1-1/z)(1-zt^{-2})} \right) &= \sum_{k=0}^{p-m} \frac{(q^{2m-2p}; q^2)_k}{(q^2; q^2)_k} (-t^{-2}q^{-2m})^k \frac{t^{-2(p-m-k)}}{1-t^{-2}} \\ &= \frac{t^{2(m-p)}}{1-t^{-2}} (-q^{-2p}; q^2)_{p-m}, \end{aligned}$$

again by the  $q$ -binomial formula [17, 1.3.14)]. We conclude that for  $p \geq m$

$$\begin{aligned} (4.4) \quad \sum_{k=-\infty}^{\infty} \frac{t^{-2k} (q^{2k+2p+2}; q^2)_\infty}{(-q^{2k}; q^2)_\infty} {}_1\varphi_1 \left( \begin{matrix} -q^{2k} \\ q^{2k+2p+2}; q^2, \frac{q^2}{t^2} \end{matrix} \right) \\ \times (q^{2k+2m+2}; q^2)_\infty {}_1\varphi_1 \left( \begin{matrix} -q^{2k} \\ q^{2k+2m+2}; q^2, \frac{q^2}{t^2} \end{matrix} \right) &= \frac{(q^2, q^2, -t^{-2}q^{-2m}; q^2)_\infty}{(-q^{-2m}; q^2)_\infty} \frac{t^{2m}}{1-t^{-2}}. \end{aligned}$$

This identity is valid under the extra assumption  $|t|^2 q^{2m} > 1$ . The right hand side of (4.4) is analytic in  $t$  for  $|t| > 1$ . Each summand on the left hand side of (4.4) is analytic in  $t$  for  $|t| > 1$ . As  $k \rightarrow \infty$  the summand behaves like  $t^{-2k}$ , and as  $k \rightarrow -\infty$  the summand behaves like  $t^{2k} q^{2(p+m)k} q^{k(k-1)}$  using (2.10), so that we obtain uniform convergence on compact sets for the left hand side of (4.4). Hence, (4.4) holds for all  $t$  with  $|t| > 1$ .

Since (4.4) gives the evaluation of the sum in (4.3) we obtain for  $p \geq m$

$$\begin{aligned} \langle Q^2 w_p^\theta(t), w_m^\theta(t) \rangle &= t^{2m} \frac{(-1, -q^2; q^2)_\infty}{(q^2 t^{-2}; q^2)_\infty} \frac{(q^2, q^2, -t^{-2}q^{-2m}; q^2)_\infty}{(-q^{-2m}; q^2)_\infty (1-t^{-2})} \\ &= \frac{(q^2, q^2, -t^{-2}, -q^2 t^2, -q^{2+2m}; q^2)_\infty}{(q^2 t^{-2}, q^2 t^{-2}, -t^2 q^{2+2m}; q^2)_\infty (1-t^{-2})} \end{aligned}$$

and using the norm of  $w_p^\theta(t)$  given in Proposition 4.1 we obtain for  $p \geq m$

$$(4.5) \quad \frac{\langle Q^2 w_p^\theta(t), w_m^\theta(t) \rangle}{\|w_p^\theta(t)\| \|w_m^\theta(t)\|} = \frac{q^{p+m}}{1-t^{-2}} \frac{(-t^2 q^{2+2p}, -q^{2+2m}; q^2)_\infty^{\frac{1}{2}}}{(-t^2 q^{2+2m}, -q^{2+2p}; q^2)_\infty^{\frac{1}{2}}}.$$

Now take  $m = p$ .  $\square$

*Remark.* By an analogous computation we obtain

$$\langle Q^2 v_p^\theta(t), w_m^\theta(t) \rangle = \frac{(-1, -q^2, q^2, q^2, -t^{-2} q^{-2m}; q^2)_\infty}{(q^2 t^{-2}, t^{-2}, -q^{-2m}; q^2)_\infty} t^{2m}, \quad p \in \mathbb{Z}_{\geq 0}, \quad m \in \mathbb{Z},$$

so that all the matrix elements of  $Q^2$  with respect to the orthogonal basis of Proposition 4.1 are known.

**Corollary 4.3.** *Let  $|t| > q^{-1}$  and let  $g$  be a bounded continuous function on the spectrum of  $\pi_1(\rho_{\infty,t})$  such that  $g(\pi_{e^{i\theta}}(\rho_{\infty,t}))Q^2$  is of trace class, then*

$$(1 - q^2) \text{Tr}_{|\ell^2(\mathbb{Z})} (g(\pi_{e^{i\theta}}(\rho_{\infty,t}))Q^2) = \frac{1}{t - t^{-1}} \int_{-t^{-1}}^{\infty(t)} g(x) d_{q^2} x = (1 - q^2) h(g(\pi(\rho_{\infty,t}))).$$

Here we use the notation for the Jackson  $q$ -integral, see [17, §1.11],  $cd > 0$ ,

$$(4.6) \quad \int_{-d}^{\infty(c)} g(x) d_q x = (1 - q)d \sum_{p=0}^{\infty} g(-dq^p) q^p + (1 - q)c \sum_{p=-\infty}^{\infty} g(cq^p) q^p.$$

Note that any finitely supported  $g$  gives a trace class operator  $g(\pi_{e^{i\theta}}(\rho_{\infty,t}))Q^2$ , and Corollary 4.3 implies that it suffices to take  $g$  satisfying  $\int_{-t^{-1}}^{\infty(t)} |g(x)| d_{q^2}(x) < \infty$ .

*Proof.* Calculate the trace with respect to the orthogonal basis of Proposition 4.1 using Lemma 4.2 for the first equality. Use Remark 2.11 to get the second equality from the first.  $\square$

**4.3 Shift operators.** In this subsection we consider the unbounded operators  $\pi_{e^{i\theta}}(\alpha_{\infty,t})$ ,  $\pi_{e^{i\theta}}(\beta_{\infty,t})$ ,  $\pi_{e^{i\theta}}(\gamma_{\infty,t})$  and  $\pi_{e^{i\theta}}(\delta_{\infty,t})$  on  $\ell^2(\mathbb{Z})$ . By Proposition 2.1 these operators are initially defined on  $\mathcal{D}(\mathbb{Z})$ , but we see, using (3.13), that these operators are defined on  $\mathcal{D}(Q) = \{\sum_k c_k e_k \in \ell^2(\mathbb{Z}) \mid \sum_k |c_k|^2 q^{-2k} < \infty\}$  and that the actions are given by the same formulas. The commutation relations of Corollary 3.9 show that we may expect that these operators act as shift operators in the basis of eigenvectors of  $\pi_{e^{i\theta}}(\rho_{\infty,t})$  of Proposition 4.1. The next proposition shows that this is the case.

**Proposition 4.4.** *Let  $|t| > 1$ , then  $v_p^\theta(t), w_p^\theta(t) \in \mathcal{D}(Q)$ . Moreover,*

$$\begin{aligned}
\pi_{e^{i\theta}}(\alpha_{\infty,t}) v_p^\theta(t) &= q^{\frac{1}{2}} e^{-i\theta} t^{-1} \frac{1 - q^{2p}}{1 - q^2 t^{-2}} v_{p-1}^\theta(tq^{-1}), \quad p \in \mathbb{Z}_{\geq 0}, \\
\pi_{e^{i\theta}}(\alpha_{\infty,t}) w_p^\theta(t) &= q^{\frac{1}{2}} e^{-i\theta} t^{-1} \frac{1 + t^2 q^{2p}}{1 - q^2 t^{-2}} w_p^\theta(tq^{-1}), \quad p \in \mathbb{Z}, \\
\pi_{e^{i\theta}}(\beta_{\infty,t}) v_p^\theta(t) &= -t^2 q^{\frac{1}{2}} e^{-i\theta} (1 - t^{-2}) v_p^\theta(tq), \quad p \in \mathbb{Z}_{\geq 0}, \\
\pi_{e^{i\theta}}(\beta_{\infty,t}) w_p^\theta(t) &= -t^2 q^{\frac{1}{2}} e^{-i\theta} (1 - t^{-2}) w_{p-1}^\theta(tq), \quad p \in \mathbb{Z}, \\
\pi_{e^{i\theta}}(\gamma_{\infty,t}) v_p^\theta(t) &= -q^{\frac{1}{2}} e^{i\theta} \frac{1 - q^{2+2p} t^{-2}}{1 - q^2 t^{-2}} v_p^\theta(tq^{-1}), \quad p \in \mathbb{Z}_{\geq 0}, \\
\pi_{e^{i\theta}}(\gamma_{\infty,t}) w_p^\theta(t) &= -q^{\frac{1}{2}} e^{i\theta} \frac{1 + q^{2+2p}}{1 - q^2 t^{-2}} w_{p+1}^\theta(tq^{-1}), \quad p \in \mathbb{Z}, \\
\pi_{e^{i\theta}}(\delta_{\infty,t}) v_p^\theta(t) &= e^{i\theta} t q^{\frac{1}{2}} (1 - t^{-2}) v_{p+1}^\theta(tq), \quad p \in \mathbb{Z}_{\geq 0}, \\
\pi_{e^{i\theta}}(\delta_{\infty,t}) w_p^\theta(t) &= e^{i\theta} t q^{\frac{1}{2}} (1 - t^{-2}) w_p^\theta(tq), \quad p \in \mathbb{Z}.
\end{aligned}$$

*Proof.* To see that for  $|t| > 1$  we have  $v_p^\theta(t), w_p^\theta(t) \in \mathcal{D}(Q)$  we proceed as in the first paragraph of the proof of Proposition 4.2. Note that the commutation relations of Corollary 3.9 suggest that the operators  $\pi_{e^{i\theta}}(\alpha_{\infty,t})$  and  $\pi_{e^{i\theta}}(\gamma_{\infty,t})$ , respectively  $\pi_{e^{i\theta}}(\beta_{\infty,t})$  and  $\pi_{e^{i\theta}}(\delta_{\infty,t})$ , map eigenvectors of  $\pi_{e^{i\theta}}(\rho_{\infty,t})$  into eigenvectors of  $\pi_{e^{i\theta}}(\rho_{\infty,t/q})$ , respectively  $\pi_{e^{i\theta}}(\rho_{\infty,tq})$ , with a possible  $q$ -shift in the eigenvalue. However, this is not sufficient since we do not have a priori estimates implying that  $\pi_{e^{i\theta}}(\alpha_{\infty,t}) v_p^\theta(t) \in \mathcal{D}(\pi_{e^{i\theta}}(\rho_{\infty,t/q}))$ . So we have to prove it in a direct manner.

Let us prove the first two statements. From Proposition 2.1 and (3.13) we get

$$\pi_{e^{i\theta}}(\alpha_{\infty,t}) e_k = q^{\frac{1}{2}} \sqrt{1 + q^{-2k}} e_{k+1} + q^{\frac{1}{2}-k} t e^{-i\theta} e_k$$

so that we get, using the notation of Proposition 4.1,

$$\pi_{e^{i\theta}}(\alpha_{\infty,t}) \sum_{k=-\infty}^{\infty} e^{ik\theta} V_k(x;t) e_k = \sum_{k=-\infty}^{\infty} q^{\frac{1}{2}} e^{i(k-1)\theta} \{ \sqrt{1 + q^{2-2k}} V_{k-1}(x;t) + q^{-k} t V_k(x;t) \} e_k,$$

for  $x = -q^{2p}/t$ ,  $p \in \mathbb{Z}_{\geq 0}$ , or  $x = tq^{2p}$ ,  $p \in \mathbb{Z}$ . We use  $(z;q)_{\infty} {}_2\varphi_1(a,b;0;q,z) = (bz;q)_{\infty} {}_1\varphi_1(b;bz;q,az)$ , see [17, (1.4.5)], to evaluate the term in curly brackets. So in terms of a  ${}_2\varphi_1$ -series we have

$$V_k(x;t) = (-q^{2-2k}; q^2)_{\infty} q^{\frac{1}{2}k(k+1)} (-t)^{-k} \frac{(-q^2 x t^{-1}; q^2)_{\infty}}{(q^2 t^{-2}; q^2)_{\infty}} {}_2\varphi_1 \left( \begin{matrix} \frac{-1}{xt}, -q^{2k} \\ 0 \end{matrix}; q^2, -q^2 \frac{x}{t} \right).$$

The contiguous relation  ${}_2\varphi_1(aq, b; 0; q, z) - {}_2\varphi_1(a, b; 0; q, z) = az(1-b){}_2\varphi_1(aq, bq; 0; q, z)$ , see [17, Exer. 1.9(ii)], gives

$$\sqrt{1 + q^{2-2k}} V_{k-1}(x;t) + q^{-k} t V_k(x;t) = t^{-1} \frac{1 + xt}{1 - q^2 t^{-2}} V_k(xq^{-1}; tq^{-1})$$

for  $|x| < |t|q^{-2}$ , and which is valid for  $x \neq 0$  by analytic continuation. This gives

$$\pi_{e^{i\theta}}(\alpha_{\infty,t}) \sum_{k=-\infty}^{\infty} e^{ik\theta} V_k(x;t) e_k = q^{\frac{1}{2}} e^{-i\theta} t^{-1} \frac{1+xt}{1-q^2 t^{-2}} \sum_{k=-\infty}^{\infty} e^{ik\theta} V_k(xq^{-1}; tq^{-1}) e_k.$$

for  $x = -q^{2p}t^{-1}$ ,  $p \in \mathbb{Z}_{\geq 0}$  and  $x = tq^{2p}$ ,  $p \in \mathbb{Z}$ , which is the desired result for  $\pi_{e^{i\theta}}(\alpha_{\infty,t})$ .

The statements for  $\pi_{e^{i\theta}}(\delta_{\infty,t})$  follow from the ones for  $\pi_{e^{i\theta}}(\alpha_{\infty,t})$  already proved and Proposition 4.1. From the factorisation in Corollary 3.9 we get  $\pi_{e^{i\theta}}(\delta_{\infty,tq^{-1}})\pi_{e^{i\theta}}(\alpha_{\infty,t}) = t\pi_{e^{i\theta}}(\rho_{\infty,t}) + 1$ . Hence, the result for  $\pi_{e^{i\theta}}(\delta_{\infty,t})$  follows for  $|t| > q^{-1}$ . The result for  $|t| > 1$  follows by component wise analytic continuation in  $t$ .

Next we consider  $\pi_{e^{i\theta}}(\gamma_{\infty,t})$ . This derivation is completely similar to the one for  $\pi_{e^{i\theta}}(\alpha_{\infty,t})$ , but now we use  ${}_2\varphi_1(a, b; c; q, z) - (1-a){}_2\varphi_1(aq, b; c; q, z) = a{}_2\varphi_1(a, b; c; q, qz)$ , which is directly verified. The statements for  $\pi_{e^{i\theta}}(\beta_{\infty,t})$  follow from the ones for  $\pi_{e^{i\theta}}(\gamma_{\infty,t})$  together with the factorisation in Corollary 3.9.  $\square$

**4.4 The Haar functional.** In Remark 2.11 we have defined  $h(T) \in [0, \infty]$  for any decomposable bounded operator acting on  $L^2(\mathbb{T}; \ell^2(\mathbb{Z}))$  such that  $T(e^{i\theta})Q^2$  is of trace class. We want to give an explicit form for the sesquilinear form  $\langle \xi_1, \xi_2 \rangle = h(\pi(\xi_2^* \xi_1))$  per bi- $K$ -type in the Cartan decomposition of Theorem 3.6 for the case  $s \rightarrow \infty$ . We have to adjust the definition of the sesquilinear form in order to apply it to sufficiently decreasing functions in Theorem 3.6. For a formal element of the form  $\xi^{(i)} = \Gamma_{l,m}^{(i)}(\infty, t)g(\rho_{\infty,t})$  for a bounded continuous function  $g$  on the spectrum of  $\pi_1(\rho_{\infty,t})$  we define the corresponding quadratic form by

$$\langle \xi^{(i)}, \xi^{(i)} \rangle = h\left(\bar{g}(\pi(\rho_{\infty,t}))\pi(\Gamma_{l,m}^{(i)}(\infty, t)^* \Gamma_{l,m}^{(i)}(\infty, t))g(\pi(\rho_{\infty,t}))\right).$$

By (4.1) and (3.20) we regard the operator in parentheses as a decomposable operator for suitable functions  $g$  and we assume it satisfies the conditions of Remark 2.11. For  $a, b \geq 0$  and  $cd > 0$  we put for functions  $f$  and  $g$

$$(4.7) \quad \langle f, g \rangle_{c,d}^{a,b} = \int_{-d}^{\infty(c)} f(x) \overline{g(x)} \frac{(-q^2 x/c, -q^2 x/d; q^2)_{\infty}}{(-q^{2+2a} x/c, -q^{2+2b} x/d; q^2)_{\infty}} dq^2 x,$$

provided that the  $q$ -integral is absolutely convergent.

**Theorem 4.5.** *Let  $g$  be a bounded continuous function on the spectrum of  $\pi_1(\rho_{\infty,t})$ , and  $l \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ ,  $m \in \{-l, -l+1, \dots, l\}$ ,  $|t| > q^{-1}$ . Assume that*

$$\bar{g}(\pi_{e^{i\theta}}(\rho_{\infty,t}))\pi_{e^{i\theta}}(\Gamma_{l,m}^{(i)}(\infty, t)^* \Gamma_{l,m}^{(i)}(\infty, t))g(\pi_{e^{i\theta}}(\rho_{\infty,t}))Q^2$$

*is of trace class on  $\ell^2(\mathbb{Z})$ . Then*

$$\begin{aligned} \langle \xi^{(1)}, \xi^{(1)} \rangle &= C_1 \langle g, g \rangle_{t, t^{-1}}^{l-m, l+m}, & \langle \xi^{(2)}, \xi^{(2)} \rangle &= C_2 \langle g, g \rangle_{t, t^{-1}q^{2l-2m}}^{l+m, l-m}, \\ \langle \xi^{(3)}, \xi^{(3)} \rangle &= C_3 \langle g, g \rangle_{tq^{2l-2m}, t^{-1}}^{l-m, l+m}, & \langle \xi^{(4)}, \xi^{(4)} \rangle &= C_4 \langle g, g \rangle_{tq^{2l+2m}, t^{-1}q^{2l-2m}}^{l+m, l-m}, \end{aligned}$$

for positive constants  $C_i$  independent of  $g$ .

*Proof.* We have  $\pi_{e^{i\theta}}(\Gamma_{l,m}^{(i)}(\infty, t)^* \Gamma_{l,m}^{(i)}(\infty, t)) = C_i p_{2l}^{(i)}(\pi_{e^{i\theta}}(\rho_{\infty, t}))$  as an unbounded operator defined on  $\mathcal{D}(\mathbb{Z})$  by (3.20) and Proposition 2.1 for explicit polynomials  $p_{2l}^{(i)}$  of degree  $2l$ . We can now use Proposition 4.1 to extend this operator to an unbounded self-adjoint operator for  $|t| > q^{-1}$ . The last statement is a consequence of Corollary 4.3.  $\square$

*Remark 4.6.* (i) The sesquilinear form in (4.7) is positive semi-definite if  $q^2 d/c < 1$ . For  $i = 1, 2$  this condition is satisfied. For  $i = 3$  we need  $q^{2-2l+2m} < t^2$  and for  $i = 4$  we need  $q^{2-4m} < t^2$  for positive semi-definiteness of the quadratic form in Theorem 4.5. To explain this phenomenon we note that in general

$$(4.8) \quad \text{Tr}|_{\ell^2(\mathbb{Z})}(\bar{g}(\pi_{e^{i\theta}}(\rho_{\infty, t}))\pi_{e^{i\theta}}(\Gamma_{l,m}^{(i)}(\infty, t)^* \Gamma_{l,m}^{(i)}(\infty, t))g(\pi_{e^{i\theta}}(\rho_{\infty, t}))Q^2)$$

is not equal to the square of the Hilbert-Schmidt norm of the operator  $S$  with  $S = \pi_{e^{i\theta}}(\Gamma_{l,m}^{(i)}(\infty, t))g(\pi_{e^{i\theta}}(\rho_{\infty, t}))Q$ . First of all, the trace is not cyclic for unbounded operators, cf. Remark 2.11. Secondly, it is not true that, under the conditions of Theorem 4.5,  $S$  extends to a bounded operator on  $\ell^2(\mathbb{Z})$ . Initially,  $S$  is only defined on  $\{v \in \mathcal{D}(\mathbb{Z}) \mid g(\pi_{e^{i\theta}}(\rho_{\infty, t}))v \in \mathcal{D}(\mathbb{Z})\}$ , which does not even have to be dense in  $\ell^2(\mathbb{Z})$ . However, if we assume that  $S$  and also  $T = SQ^{-1}$  are defined on finite linear combinations of the orthogonal basis of  $\ell^2(\mathbb{Z})$  given in Proposition 4.1, and then have an extension to a bounded operator on  $\ell^2(\mathbb{Z})$ , then (4.8) indeed equals the square of the Hilbert-Schmidt norm of  $S$ , and positivity follows. Let us consider the case  $i = 3$ , the other cases are similar. By Proposition 4.1, Proposition 4.4 and (3.16) for  $s = \infty$ , we see that we can calculate  $Tv_p^\theta(t)$  and  $Tw_p^\theta(t)$  explicitly for  $|t| > q^{1-2l}$ . From this we can determine conditions on  $g$  that imply that  $T$  extends to a bounded operator on  $\ell^2(\mathbb{Z})$ , e.g. it suffices to consider compactly supported  $g$  with  $0 \notin \text{supp}(g)$ . For such  $g$  we can also extend  $S$  to a bounded operator on  $\ell^2(\mathbb{Z})$ , since we can estimate the growth of  $\|Qv_p^\theta(t)\|$  and  $\|Qw_p^\theta(t)\|$  from Lemma 4.2. Note that  $|t| > q^{1-2l}$  implies  $|t|^2 > q^{2-2l+2m}$ , since  $|m| \leq l$ .

(ii) In case  $q^2 d/c < 1$ ,  $\langle \cdot, \cdot \rangle_{c,d}^{a,b}$  gives an inner product. The corresponding Hilbert space is a weighted  $L^2$ -space, on which the big  $q$ -Jacobi function transform lives, see [28] and §6 for a quantum group theoretic interpretation.

*Remark 4.7.* It would be desirable to interpret the elements  $\pi(\Gamma_{l,m}^{(i)}(\infty, t))g(\pi(\rho_{\infty, t}))$  in Theorem 4.5 in terms of affiliated elements for the  $C^*$ -algebra  $\pi(C_0(X) \times_\tau \mathbb{Z})$  or more generally, in terms of regular operators for Hilbert  $C^*$ -modules, see Woronowicz [62] for the notion of affiliated elements and [39], [36] for regular operators. This would give rise to an interpretation of  $\Gamma_{l,m}^{(i)}(\infty, t)g(\rho_{\infty, t})$  as a uniquely defined element affiliated to the  $C^*$ -algebra  $C_0(X) \times_\tau \mathbb{Z}$ , see Kustermans [36]. In general this seems not to be possible due to the fact that the density requirements in either the definition of affiliated element in [62] or in the definition of regular operator in [39] is not met. Let  $f \in C_c(\mathbb{Z}; C(X))$  such that it is supported in precisely one point. It is straightforward to check that multiplication by  $\pi(f)$  is a regular operator of the  $C^*$ -algebra  $\pi(C_0(X) \times_\tau \mathbb{Z})$  viewed as a Hilbert  $C^*$ -module over itself, see Lance [39, Ch. 9] and Woronowicz [62, §3.C]. However, it is not clear if this



remains true for  $f \in C_c(\mathbb{Z}; C(X))$  supported in more than just one point, such as for  $f$  corresponding to  $\pi(\rho_{\infty,t})$ .

Moreover, it is also unclear that for functions  $g \in C_0(\mathbb{R})$  the operator of multiplication by  $g(\pi(\rho_{\infty,t}))$  is a multiplier of the  $C^*$ -algebra  $\pi(C_0(X) \times_{\tau} \mathbb{Z})$ . The problem is that it is not clear if this multiplication operator preserves the  $C^*$ -algebra  $\pi(C_0(X) \times_{\tau} \mathbb{Z})$ . However, in this particular case  $s = \infty$ , estimates can be obtained that show that  $g(\pi(\rho_{\infty,t})) \in \pi(C_0(X) \times_{\tau} \mathbb{Z})$  for  $g$  finitely supported on the spectrum.

## 5. THE HAAR FUNCTIONAL ON THE ALGEBRA GENERATED BY $\rho_{s,t}$

In this section we calculate explicitly the Haar functional related to the Cartan decomposition of Theorem 3.6. The result is given in terms of a non-compact analogue of the Askey-Wilson measure, and it is obtained using the spectral analysis of  $\pi_{e^{i\theta}}(\rho_{s,t})$  and (2.17). This operator is considered in two invariant complementary subspaces  $V^{\theta}(t)$  and  $W^{\theta}(t)$  of  $\ell^2(\mathbb{Z})$ . The spectral decomposition of  $\pi_{e^{i\theta}}(\rho_{s,t})$  on  $V^{\theta}(t)$  is obtained using orthogonal polynomials and is analogous to [31, §6]. On  $W^{\theta}(t)$  the spectral analysis is related to the little  $q$ -Jacobi function transform. Matching these two results involves non-trivial summation formulas for basic hypergeometric series. The main new summation formula has been proved by Mizan Rahman, and its proof is presented in Appendix B. Recall the basic assumption that  $s$  and  $t$  are real parameters, and we also assume that  $|s|, |t| > q^{-1}$ .

**5.1 Spectral analysis of  $\pi_{e^{i\theta}}(\rho_{s,t})|_{V^{\theta}(t)}$ .** In this subsection we calculate the spectral measure for  $\pi_{e^{i\theta}}(\rho_{s,t})|_{V^{\theta}(t)}$ , which is a bounded operator that can be viewed as a Jacobi matrix. This enables us to link it to the Al-Salam and Chihara polynomials. The analysis in this part follows [31, §6].

The operator  $\pi_{e^{i\theta}}(\rho_{s,t})$  is an unbounded five-term recurrence operator in the standard basis  $\{e_k \mid k \in \mathbb{Z}\}$  of  $\ell^2(\mathbb{Z})$  densely defined on  $\mathcal{D}(\mathbb{Z})$  by Proposition 2.1 and (3.11). We can extend the domain of  $\pi_{e^{i\theta}}(\rho_{s,t})$  to  $\mathcal{D}(Q^2)$ , since  $\rho_{s,t}$  consists of quadratic elements in the generators  $\alpha$  and  $\gamma$ . Since  $v_p^{\theta}(t) \in \mathcal{D}(Q^2)$  for  $|t| > q^{-1}$  we see that  $\pi_{e^{i\theta}}(\rho_{s,t})v_p^{\theta}(t)$  is well-defined. It follows from (3.21) and Proposition 4.4 that  $\pi_{e^{i\theta}}(\rho_{s,t})$  is a three-term recurrence operator in the basis  $v_p^{\theta}(t)$ ,  $p \in \mathbb{Z}_{\geq 0}$ ;

$$(5.1) \quad 2\pi_{e^{i\theta}}(\rho_{s,t})v_p^{\theta}(t) = -qe^{2i\theta}(1 - q^{2+2p}t^{-2})v_{p+1}^{\theta}(t) + q^{1+2p}t^{-1}(s + s^{-1})v_p^{\theta}(t) - q^{-1}e^{-2i\theta}(1 - q^{2p})v_{p-1}^{\theta}(t).$$

Note that  $\pi_{e^{i\theta}}(\rho_{s,t})|_{V^{\theta}(t)}$  is a bounded operator. By going over to the orthonormal basis  $f_p = (-e^{2i\theta})^p v_p^{\theta}(t) / \|v_p^{\theta}(t)\|$ ,  $p \in \mathbb{Z}_{\geq 0}$ , see Proposition 4.1, we obtain

$$(5.2) \quad \begin{aligned} 2\pi_{e^{i\theta}}(\rho_{s,t})f_p &= a_{p+1}f_{p+1} + b_p f_p + a_p f_{p-1}, \quad p \in \mathbb{Z}_{\geq 0}, \\ a_p &= \sqrt{(1 - q^{2p}t^{-2})(1 - q^{2p})}, \quad b_p = q^{1+2p}t^{-1}(s + s^{-1}), \end{aligned}$$

which is, by Favard's Theorem, a three-term recurrence for orthonormal polynomials since  $|t| > 1$ . Note that  $a_0 = 0$ , so that (5.2) is a well-defined operator. The spectral measure can be determined completely in terms of the orthogonality measure of the corresponding

orthonormal polynomials, see e.g. [1], [30], [31], [54]. The polynomials can be identified with the Al-Salam and Chihara polynomials.

We recall that the Al-Salam and Chihara polynomials, originally introduced by Al-Salam and Chihara in [2], are orthogonal polynomials with respect to an absolutely continuous measure on  $[-1, 1]$  plus a finite set, possibly empty, of discrete mass points as established by Askey and Ismail [3]. The Al-Salam and Chihara polynomials are a subclass of the Askey-Wilson polynomials by setting two parameters of the four parameters of the Askey-Wilson polynomials equal to zero, see Askey and Wilson [5], or [17, Ch. 7].

The Al-Salam and Chihara polynomials are defined by

$$(5.3) \quad s_m(\cos \psi; a, b|q) = a^{-m} (ab; q)_m {}_3\varphi_2 \left( \begin{matrix} q^{-m}, ae^{i\psi}, ae^{-i\psi} \\ ab, 0 \end{matrix}; q, q \right).$$

Let  $S_m(x; a, b|q) = s_m(x; a, b|q) / \sqrt{(q, ab; q)_m}$  denote the orthonormal Al-Salam and Chihara polynomials, which satisfy the three-term recurrence relation

$$(5.4) \quad \begin{aligned} 2x S_n(x) &= a_{n+1} S_{n+1}(x) + q^n(a+b) S_n(x) + a_n S_{n-1}(x), \\ a_n &= \sqrt{(1 - abq^{n-1})(1 - q^n)}. \end{aligned}$$

We assume  $ab < 1$ , so that  $a_n > 0$ . Since the coefficients  $a_n$  and  $b_n$  are bounded, the corresponding moment problem is determined and the orthonormal Al-Salam and Chihara polynomials form an orthonormal basis of  $L^2(\mathbb{R}, dm(\cdot; a, b|q^2))$ , with  $dm(\cdot; a, b|q^2)$  the normalised orthogonality measure. The explicit form of the orthogonality measure is originally obtained by Askey and Ismail [3], and it is a special case of the Askey-Wilson measure. Since the Askey-Wilson measure is needed in the next subsection we recall it here, see [5, §2], [17, Ch. 6]:

$$(5.5) \quad \int_{\mathbb{R}} p(x) dm(x; a, b, c, d|q) = \frac{1}{h_0 2\pi} \int_0^\pi p(\cos \theta) w(e^{i\theta}) d\theta + \frac{1}{h_0} \sum_k p(x_k) w_k.$$

Here we use the notation  $w(z) = w(z; a, b, c, d|q)$ ,  $h_0 = h_0(a, b, c, d|q)$  and

$$(5.6) \quad \begin{aligned} h_0(a, b, c, d|q) &= \frac{(abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty}, \\ w(z; a, b, c, d|q) &= \frac{(z^2, z^{-2}; q)_\infty}{(az, a/z, bz, b/z, cz, c/z, dz, d/z; q)_\infty}, \end{aligned}$$

and we suppose that  $a, b, c$  and  $d$  are real or  $\bar{a} = b$ , and  $c, d \in \mathbb{R}$  and such that all pairwise products are less than 1. The sum in (5.5) is over the points  $x_k$  of the form  $\mu(eq^k) = (eq^k + e^{-1}q^{-k})/2$  with  $e$  any of the parameters  $a, b, c$  or  $d$  whose absolute value is larger than one and such that  $|eq^k| > 1$ ,  $k \in \mathbb{Z}_{\geq 0}$ . The corresponding mass  $w_k$  is the residue of  $z \mapsto w(z)/z$  at  $z = eq^k$ . The value of  $w_k$  in case  $e = a$  is given in [5, (2.10)], [17, (6.6.12)]. Explicitly,

$$(5.7) \quad \begin{aligned} w_k(a; b, c, d|q) &= \frac{(a^{-2}; q)_\infty}{(q, ab, b/a, ac, c/a, ad, d/a; q)_\infty} \\ &\quad \times \frac{(1 - a^2 q^{2k})}{(1 - a^2)} \frac{(a^2, ab, ac, ad; q)_k}{(q, aq/b, aq/c, aq/d; q)_k} \left( \frac{q}{abcd} \right)^k. \end{aligned}$$

The orthogonality measure for the Al-Salam and Chihara polynomials is obtained by taking  $c = d = 0$  in (5.5), so  $dm(\cdot; a, b|q) = dm(\cdot; a, b, 0, 0|q)$ .

Now compare (5.2) with (5.4) in base  $q^2$  with  $a$  and  $b$  replaced by  $qst^{-1}$  and  $qs^{-1}t^{-1}$ . This shows that we can realise  $\pi_{e^{i\theta}}(\rho_{s,t})|_{V^\theta(t)}$  as a multiplication operator on the weighted  $L^2$ -space corresponding to the orthogonality measure  $dm(\cdot; qst^{-1}, qs^{-1}t^{-1}|q^2)$  using the unitary isomorphism  $V^\theta(t) \rightarrow L^2(\mathbb{R}, dm(\cdot; qst^{-1}, qs^{-1}t^{-1}|q^2))$  mapping  $f_p$  to the corresponding  $p$ -th orthonormal polynomial  $S_p(\cdot; qst^{-1}, qs^{-1}t^{-1}|q^2)$ , see e.g. Akhiezer [1, Ch. 1], Simon [54]. This proves the following proposition.

**Proposition 5.1.** *Let  $s, t \in \mathbb{R}$  with  $|s| > 1$ ,  $|t| > q^{-1}$ . The spectrum of the bounded self-adjoint operator  $\pi_{e^{i\theta}}(\rho_{s,t})|_{V^\theta(t)}$  consists of the continuous spectrum  $[-1, 1]$  and the finite discrete spectrum, possibly empty,  $\{q^{1+2k}st^{-1} \mid |q^{1+2k}st^{-1}| > 1, k \in \mathbb{Z}_{\geq 0}\}$ . Explicitly, with  $f_p = (-e^{2i\theta})^p v_p^\theta(t)/\|v_p^\theta(t)\|$ ,*

$$\langle \pi_{e^{i\theta}}(\rho_{s,t})f_n, f_m \rangle = \int_{\mathbb{R}} x(S_n S_m)(x; qst^{-1}, qs^{-1}t^{-1}|q^2) dm(x; qst^{-1}, qs^{-1}t^{-1}|q^2).$$

**Proposition 5.2.** *Let  $s, t \in \mathbb{R}$  with  $|s|, |t| > q^{-1}$  and let  $P$  be the orthogonal projection onto  $V^\theta(t)$  along the decomposition  $\ell^2(\mathbb{Z}) = V^\theta(t) \oplus W^\theta(t)$ . Then  $PQ^2|_{V^\theta(t)}: V^\theta(t) \rightarrow V^\theta(t)$  is bounded. Let  $f$  be a continuous function on the spectrum of  $\pi_{e^{i\theta}}(\rho_{s,t})|_{V^\theta(t)}$ , and assume that  $f(\pi_{e^{i\theta}}(\rho_{s,t})|_{V^\theta(t)})PQ^2$  is of trace class on  $V^\theta(t)$ . Then its trace is integrable over  $[0, 2\pi]$  as function of  $\theta$  and*

$$\begin{aligned} & \frac{1}{2\pi} \int_0^{2\pi} \text{Tr}|_{V^\theta(t)}(f(\pi_{e^{i\theta}}(\rho_{s,t})|_{V^\theta(t)})PQ^2) d\theta = \\ & \frac{1}{2\pi} \int_0^\pi f(\cos \theta) \frac{(1 - q^2/t^2)(1 - e^{\pm 2i\theta})}{(t^2 - 1)(1 - \frac{qs}{t}e^{\pm i\theta})(1 - \frac{q}{st}e^{\pm i\theta})} {}_8W_7\left(\frac{q^2}{t^2}; q^2, \frac{qs}{t}e^{\pm i\theta}, \frac{q}{st}e^{\pm i\theta}; q^2, q^2\right) d\theta \\ & + \sum_{k \in \mathbb{Z}_{\geq 0}, |q^{1+2k}s/t| > 1} \frac{w_k(qs/t, q/st, qt/s, qst|q^2)}{h_0(qs/t, q/st, qt/s, qst|q^2)} \frac{-1}{1 - q^2} f(\mu(q^{1+2k}s/t)), \end{aligned}$$

where  $\mu(z) = \frac{1}{2}(z + z^{-1})$ . The  $\pm$ -sign means that we have to take all possibilities.

The positivity of the weight for the discrete mass points in Proposition 5.2 follows from

$$\frac{-1}{1 - q^2} \frac{w_k(qs/t, q/st, qt/s, qst|q^2)}{h_0(qs/t, q/st, qt/s, qst|q^2)} = \frac{(q^{2+4k}s^2 - t^2)q^{-2-2k}}{(s^2 - 1)(t^2 - 1)},$$

using (5.6), (5.7). For  $|q^{1+2k}s/t| > 1$  this is positive.

Note that in the  ${}_8W_7$ -series, see §1, a lot of cancellation occurs,

$$\begin{aligned} & {}_8W_7\left(\frac{q^2}{t^2}; q^2, \frac{qs}{t}e^{i\theta}, \frac{qs}{t}e^{-i\theta}, \frac{q}{st}e^{i\theta}, \frac{q}{st}e^{-i\theta}; q^2, q^2\right) = \\ & \sum_{k=0}^{\infty} \frac{1 - t^{-2}q^{2+4k}}{1 - t^{-2}q^2} \frac{(1 - \frac{qs}{t}e^{i\theta})(1 - \frac{qs}{t}e^{-i\theta})(1 - \frac{q}{st}e^{i\theta})(1 - \frac{q}{st}e^{-i\theta})}{(1 - \frac{q^{2k+1}s}{t}e^{i\theta})(1 - \frac{q^{2k+1}s}{t}e^{-i\theta})(1 - \frac{q^{2k+1}}{st}e^{i\theta})(1 - \frac{q^{2k+1}}{st}e^{-i\theta})} q^{2k}. \end{aligned}$$

This can be used to integrate the function  $f = 1$  explicitly over the interval  $[0, \pi]$  in Proposition 5.2.

*Proof of Proposition 5.2.* Since (4.2) implies

$$\left| \frac{\langle Q^2 v_n^\theta(t), v_m^\theta(t) \rangle}{\|v_n^\theta\| \|v_m^\theta\|} \right| \leq C q^{m+n}, \quad n, m \in \mathbb{Z}_{\geq 0},$$

using that  $Q^2$  is self-adjoint and that  $v_n^\theta(t) \in \mathcal{D}(Q^2)$  for  $|t| > q^{-1}$ , it follows that  $PQ^2|_{V^\theta(t)}$  is bounded. The rest of the proof of Proposition 5.2 is completely analogous to the proof of [31, Prop. 6.3], see also the proof of Proposition 5.4. Use Proposition 5.1 and (4.2) to calculate the trace formally as a double sum involving an integral of a product of  $f$  and two Al-Salam and Chihara polynomials. Since the  $\theta$ -dependence in  $\text{Tr}|_{V^\theta(t)}(f(\pi_{e^{i\theta}}(\rho_{s,t}))Q^2)$  is easy, integration over  $[0, 2\pi]$  reduces to a single sum. Interchanging summation and integration gives an integral involving the Poisson kernel for the Al-Salam and Chihara polynomials. This can be justified using the same estimate as in the beginning of the proof and the asymptotics for the Al-Salam and Chihara polynomials, see [3, §3.3]. The Poisson kernel for the Al-Salam and Chihara polynomials is given in terms of a very-well-poised  ${}_8\varphi_7$ -series by Askey, Rahman and Suslov [4, (14.8)], see also [20, §4] and [58] for other derivations. The very-well-poised  ${}_8\varphi_7$ -series is summable for points in the discrete spectrum. See [31, §6] for details.  $\square$

**5.2 Spectral analysis of  $\pi_{e^{i\theta}}(\rho_{s,t})|_{W^\theta(t)}$ .** In this subsection we calculate the spectral measure for  $\pi_{e^{i\theta}}(\rho_{s,t})|_{W^\theta(t)}$ , which is an unbounded operator that can be viewed as doubly infinite Jacobi matrix. The operator has been studied by Takechi [21], Takechi, Masuda and Ueno [22] in connection with the spherical Fourier transform on the quantum  $SU(1, 1)$ , i.e. corresponding to the Cartan decomposition of Theorem 3.6 for the case  $(s, t) = (\infty, \infty)$ . This is the little  $q$ -Jacobi function transform, and it is discussed in Appendix A.

As in the previous subsection, cf. (5.1), we can apply  $\pi_{e^{i\theta}}(\rho_{s,t})$  to  $w_p^\theta(t)$  for  $|t| > q^{-1}$ . From (3.21) and Proposition 4.4 we see that  $\pi_{e^{i\theta}}(\rho_{s,t})$  is a three-term recurrence operator in the basis  $w_p^\theta(t)$ ,  $p \in \mathbb{Z}$ , of  $W^\theta(t)$ ;

$$(5.8) \quad \begin{aligned} 2\pi_{e^{i\theta}}(\rho_{s,t})w_p^\theta(t) &= -qe^{2i\theta}(1 + q^{2+2p})w_{p+1}^\theta(t) \\ &\quad - q^{1+2p}t(s + s^{-1})w_p^\theta(t) - q^{-1}e^{-2i\theta}(1 + t^2q^{2p})w_{p-1}^\theta(t). \end{aligned}$$

By going over to the orthonormal basis  $f_{-p} = (-e^{2i\theta})^p w_p^\theta(t) / \|w_p^\theta(t)\|$ ,  $p \in \mathbb{Z}$ , see Proposition 4.1, we obtain

$$(5.9) \quad \begin{aligned} 2\pi_{e^{i\theta}}(\rho_{s,t})f_p &= a_{p+1}f_{p+1} + b_p f_p + a_p f_{p-1}, \quad p \in \mathbb{Z}, \\ a_p &= \sqrt{(1 + q^{2-2p}t^2)(1 + q^{2-2p})}, \quad b_p = -q^{1-2p}t(s + s^{-1}). \end{aligned}$$

This is an unbounded symmetric operator that has been studied in [21], [22], see also Appendix A, Theorem A.5. So the spectral measure of the operator  $\pi_{e^{i\theta}}(\rho_{s,t})|_{W^\theta(t)}$  is determined in terms of little  $q$ -Jacobi functions.

Put

$$\phi_n(x; s, t|q^2) = {}_2\varphi_1 \left( \begin{matrix} qs^{-1}t^{-1}z, qs^{-1}t^{-1}z^{-1} \\ q^2s^{-2} \end{matrix}; q^2, -q^{2n} \right), \quad x = \mu(z) = \frac{1}{2}(z + z^{-1}),$$

for the little  $q$ -Jacobi function adapted to our situation, see Appendix A for its definition in case  $n \leq 0$ . Put, cf. (A.13),

$$(5.10) \quad w_n = (qs^{-1}t^{-1})^n \sqrt{\frac{(-q^{2-2n}t^2; q^2)_\infty}{(-q^{2-2n}; q^2)_\infty}},$$

so that the little  $q$ -Jacobi function transform is given by

$$(\mathcal{G}u)(x) = \sum_{n=-\infty}^{\infty} w_n \phi_n(x; s, t|q^2) u_n, \quad u = \sum_{n=-\infty}^{\infty} u_n f_n \in W^\theta(t),$$

initially defined for finite sums and extended to  $W^\theta(t)$  by continuity, see Appendix A. In order to describe the spectral measure we introduce the following measure

$$(5.11) \quad \int_{\mathbb{R}} f(x) d\nu(x; a, b; d|q) = h_0(a, b, q/d, d|q) \int_{\mathbb{R}} f(x) dm(x; a, b, q/d, d|q) \\ + \sum_{k \in \mathbb{N}} f(\mu(dq^{-k})) \text{Res}_{z=dq^{-k}} \frac{w(z; a, b, q/d, d|q)}{z}$$

using the notation of (5.6). Observe that

$$(5.12) \quad \text{Res}_{z=dq^{-k}} \frac{w(z; a, b, q/d, d|q)}{z} = \frac{-d^{2(k-1)} q^{-k(k-1)} (1 - d^2 q^{-2k})}{(q, q, adq^{-k}, bdq^{-k}, aq^k/d, bq^k/d; q)_\infty}$$

by a straightforward calculation, which equals  $-w_{k-1}(\frac{q}{d}; a, b, d|q)$  using (5.7). It follows that this measure is supported on  $[-1, 1]$  plus a finite, possibly empty, set of discrete mass points of the form  $\{\mu(eq^k) \mid k \in \mathbb{Z}_{\geq 0}, |eq^k| > 1\}$ , where  $e$  is  $a$  or  $b$ , plus an infinite set of discrete mass points of the form  $\{\mu(dq^{-k}) \mid k \in \mathbb{Z}, |dq^{-k}| > 1\}$ .

*Remark.* The measure  $d\nu(x; a, b; d|q)$  defined in (5.11) is positive for  $ab < 1$ ,  $ad < 0$ ,  $bd < 0$  or for  $a, b$  in complex conjugate pair with  $ab < 1$ , and has unbounded support. The measure in (5.11) can be obtained from the standard Askey-Wilson measure by a limiting procedure; consider the Askey-Wilson measure with parameters  $a, b, cq^l$  and  $dq^{-l}$ , and let  $l \rightarrow \infty$ . The parameter  $c$  disappears in the limit. Then we formally obtain the measure of (5.11), and in this way we formally obtain the little  $q$ -Jacobi function transform as a limit case of the orthogonality relations for the Askey-Wilson polynomials. In the corresponding quantum group theoretic setting this corresponds to the limit transition of the compact quantum  $SU(2)$  group to the quantum  $E(2)$  group of orientation and distance preserving motions of the Euclidean plane. In that case, (5.11) gives an expression for the Haar functional on a certain subalgebra, and  $\mathbb{Z}$  labels the representations of the quantum group of plane motions, see [23, Ch. 3].

The results of [21] on the little  $q$ -Jacobi function transform imply the following proposition, see also Theorem A.5, case (3).

**Proposition 5.3.** *Let  $s, t \in \mathbb{R}$  with  $|s|, |t| > q^{-1}$ . The operator  $\pi_{e^{i\theta}}(\rho_{s,t})|_{W^\theta(t)}$  is essentially self-adjoint and its spectral decomposition is given by*

$$\langle \pi_{e^{i\theta}}(\rho_{s,t})f_n, f_m \rangle = C \int_{\mathbb{R}} x w_n w_m (\phi_n \phi_m)(x; s, t|q^2) d\nu(x; q/st, qt/s; -qst|q^2)$$

with  $C = (q^2 s^{-2}, -1, -q^2; q^2)_\infty^2$ . The support of  $d\nu(\cdot; q/st, qt/s; -qst|q^2)$  is the spectrum of  $\pi_{e^{i\theta}}(\rho_{s,t})|_{W^\theta(t)}$ .

**Proposition 5.4.** *Let  $|s|, |t| > q^{-1}$ , and assume  $s^2 t^{\pm 2} \notin q^{2\mathbb{Z}}$ . Let  $f$  be a continuous, compactly supported function on the spectrum of  $\pi_{e^{i\theta}}(\rho_{s,t})|_{W^\theta(t)}$ , integrable with respect to the measure  $d\nu(\cdot; qs^{-1}t^{-1}, qts^{-1}; -qst|q^2)$ , and such that  $f(\pi_{e^{i\theta}}(\rho_{s,t})|_{W^\theta(t)})(1-P)Q^2$ , with  $P$  as in Proposition 5.2, is of trace class on  $W^\theta(t)$ . Then*

$$\theta \mapsto \text{Tr}|_{W^\theta(t)}(f(\pi_{e^{i\theta}}(\rho_{s,t})|_{W^\theta(t)})(1-P)Q^2)$$

is integrable over  $[0, 2\pi]$  and

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \text{Tr}|_{W^\theta(t)}(f(\pi_{e^{i\theta}}(\rho_{s,t})|_{W^\theta(t)})(1-P)Q^2) d\theta = \\ \frac{(q^2 s^{-2}, -1, -q^2; q^2)_\infty^2}{1-t^{-2}} \int_{\mathbb{R}} f(x) R_{q^{-2}}(x; s, t|q^2) d\nu(x; qs^{-1}t^{-1}, qts^{-1}; -qst|q^2) \end{aligned}$$

where  $R_u(x; s, t|q^2) = \sum_{n=-\infty}^{\infty} u^n w_n^2 |\phi_n(x; s, t|q^2)|^2$  is absolutely convergent for  $u = q^{-2}$ , uniformly for  $x$  in compacta of the support of  $d\nu(\cdot; qs^{-1}t^{-1}, qts^{-1}; -qst|q^2)$ .

*Proof.* Using  $f_{-p} = (-e^{2i\theta})^p w_p^\theta(t) / \|w_p^\theta(t)\|$  and the spectral decomposition of Proposition 5.3 we calculate the trace as

$$\begin{aligned} (5.13) \quad \text{Tr}|_{W^\theta(t)}(f(\pi_{e^{i\theta}}(\rho_{s,t}))Q^2) &= (q^2 s^{-2}, -1, -q^2; q^2)_\infty^2 \sum_{n,m=-\infty}^{\infty} (-e^{2i\theta})^{(m-n)} \\ &\times \frac{\langle Q^2 w_{-n}^\theta(t), w_{-m}^\theta(t) \rangle}{\|w_{-n}^\theta(t)\| \|w_{-m}^\theta(t)\|} w_n w_m \int_{\mathbb{R}} f(x) \phi_n(x) \phi_m(x) d\nu(x; qs^{-1}t^{-1}, qts^{-1}; -qst|q^2). \end{aligned}$$

Note that by symmetry in  $n$  and  $m$ , since  $Q^2$  is self-adjoint and  $w_n^\theta(t) \in \mathcal{D}(Q^2)$  for  $|t| > q^{-1}$ , we may restrict to  $n \leq m$ . We estimate the double sum

$$\sum_{n=-\infty}^{\infty} \sum_{m=n}^{\infty} \left| \frac{\langle Q^2 w_{-n}^\theta(t), w_{-m}^\theta(t) \rangle}{\|w_{-n}^\theta(t)\| \|w_{-m}^\theta(t)\|} w_n w_m \phi_n(x) \phi_m(x) \right|$$

and this suffices for most points of the spectrum. The weight function is only needed for the cases  $x = \pm 1$ .

Using (4.5) we obtain

$$\left| \frac{\langle Q^2 w_{-n}^\theta(t), w_{-m}^\theta(t) \rangle}{\|w_{-n}^\theta(t)\| \|w_{-m}^\theta(t)\|} w_n w_m \right| = \frac{|st|^{-n-m} (-t^2 q^{2-2n}; q^2)_\infty}{1-t^{-2} (-q^{2-2n}; q^2)_\infty} \leq \begin{cases} C |ts|^{-n-m}, & n \leq 0, \\ C |t/s|^n |st|^{-m}, & n \geq 0, \end{cases}$$

using (5.10) and (2.9). From the definition of  $\phi_n(\cdot; s, t|q^2)$  it is immediate that, for  $x$  in compact subsets of  $\text{supp}(d\nu)$ ,  $|\phi_n(\cdot; s, t|q^2)|$  is uniformly bounded for  $n \geq 0$ . Hence, the  $\sum_{n=0}^{\infty} \sum_{m=n}^{\infty}$  part of the double sum can be majorised by  $C \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} |st|^{-m} |t/s|^n < \infty$ , uniformly for  $x$  in compact subsets of  $\text{supp}(d\nu)$ .

It remains to consider  $\sum_{n=-\infty}^0 \sum_{m=n}^{\infty}$ . For this we have to estimate  $\phi_n(x; s, t|q^2)$  for  $n \leq 0$  for  $x$  in the support of the measure. Using the  $c$ -function expansion, see [17, (4.3.2)] or (A.10), or e.g. [21], [22], [28], we find

$$(5.14) \quad \begin{aligned} \phi_n\left(\frac{1}{2}(z + z^{-1}); s, t|q^2\right) &= c(z)\Phi_n(z; s, t; q^2) + c(z^{-1})\Phi_n(z^{-1}; s, t; q^2), \\ c(z) &= \frac{(qt/sz, q/stz, -qz/st, -qst/z; q^2)_{\infty}}{(z^{-2}, q^2/s^2, -1, -q^2; q^2)_{\infty}}, \\ \Phi_n(z; s, t|q^2) &= \left(\frac{qz}{st}\right)^{-n} {}_2\varphi_1\left(\begin{matrix} qz/st, qzs/t \\ q^2z^2 \end{matrix}; q^2, -t^2q^{2-2n}\right) \end{aligned}$$

for  $z^2 \notin q^{2\mathbb{Z}}$ . See also §A.2 for the definition of  $\Phi_n(z; s, t|q^2)$  in case  $|t^2q^{2-2n}| \geq 1$ . For  $n$  sufficiently negative we can estimate  $\Phi_n(e^{i\psi}; s, t|q^2)$  by  $|st/q|^n$  times a constant for  $0 \leq \psi \leq \pi$  by continuity. Hence, for  $n, m \leq 0$  we get, using that  $(e^{2i\psi}, e^{-2i\psi}; q^2)_{\infty}$  is part of the weight function of the measure,

$$|(e^{2i\psi}, e^{-2i\psi}; q^2)_{\infty} \phi_n(\cos \psi; s, t|q^2) \phi_m(\cos \psi; s, t|q^2)| \leq C |st/q|^{n+m}.$$

So, on the interval  $[-1, 1]$  the sum  $\sum_{n=-\infty}^0 \sum_{m=n}^{\infty}$ , after multiplication by  $(e^{\pm 2i\psi}; q^2)_{\infty}$ , is estimated by

$$\sum_{n=-\infty}^0 q^{-n} \left( \sum_{m=n}^0 q^{-m} + \sum_{m=0}^{\infty} |st|^{-m} \right) < \infty.$$

This deals with the convergence of (5.13) on the absolutely continuous part.

For the discrete part we observe that for  $z_k = tq^{1+2k}/s$ ,  $k \in \mathbb{Z}_{\geq 0}$ ,  $|z_k| > 1$ , or for  $z_k = -stq^{-1-2k}$ ,  $k \in \mathbb{Z}$ ,  $|z_k| > 1$ , we see that  $c(z_k) = 0$  and by (5.14)  $|\phi_n(\mu(z_k))| \leq C |q/stz_k|^{-n}$  for  $n \leq 0$ . This estimate then shows that the double sum is absolutely convergent, and uniform for  $x$  in compact subsets of the support of the measure. Note that we have used  $s^2/t^2, s^2t^2 \notin q^{2\mathbb{Z}}$  to avoid zeroes in the denominator of the  $c$ -function of (5.14) at  $z = z_k$ .

The above proves that the double sum in (5.13) is absolutely convergent. Since the estimates are uniformly in  $\theta$ , we see that  $\theta \mapsto \text{Tr}|_{W^{\theta}(t)}(f(\pi_{e^{i\theta}}(\rho_{s,t})|_{W^{\theta}(t)})(1-P)Q^2)$  is integrable over  $[0, 2\pi]$  and moreover that we may integrate term by term in (5.13) and interchange summation. This gives using Lemma 4.2,

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \text{Tr}|_{W^{\theta}(t)}(f(\pi_{e^{i\theta}}(\rho_{s,t})|_{W^{\theta}(t)})(1-P)Q^2) d\theta &= \frac{(q^2s^{-2}, -1, -q^2; q^2)_{\infty}^2}{(1-t^{-2})} \\ &\times \int_{\mathbb{R}} f(x) \sum_{n=-\infty}^{\infty} q^{-2n} w_n^2 |\phi_n(x; s, t|q^2)|^2 d\nu(x; qs^{-1}t^{-1}, qts^{-1}; -qst|q^2) \end{aligned}$$

which is the expression stated.

Using the explicit form of  $R_u(x; s, t|q^2)$  and the estimates already in use on the little  $q$ -Jacobi functions we immediately obtain that the sum in  $R_{q^{-2}}$  is absolutely convergent both for  $x$  in the absolutely continuous part and for  $x$  in the discrete part of the measure  $d\nu(\cdot; qs^{-1}t^{-1}, qts^{-1}; -qst|q^2)$ , and even uniformly for  $x$  in compact subsets of the support of the measure.  $\square$

**5.3 The trace of  $f(\pi_{e^{i\theta}}(\rho_{s,t}))Q^2$ .** We calculate the trace of  $f(\pi_{e^{i\theta}}(\rho_{s,t}))Q^2$  in this subsection and we integrate the result over  $[0, 2\pi]$ . This gives the Haar functional on  $f(\rho_{s,t})$ , see Remark 2.11, as an explicit Askey-Wilson type integral with unbounded support of the form (5.11).

Since  $f(\pi_{e^{i\theta}}(\rho_{s,t}))$  preserves the decomposition  $\ell^2(\mathbb{Z}) = V^\theta(t) \oplus W^\theta(t)$  arising from Proposition 4.1 we have that

$$\text{Tr}|_{\ell^2(\mathbb{Z})} f(\pi_{e^{i\theta}}(\rho_{s,t}))Q^2 = \text{Tr}|_{V^\theta(t)} f(\pi_{e^{i\theta}}(\rho_{s,t}))Q^2 + \text{Tr}|_{W^\theta(t)} f(\pi_{e^{i\theta}}(\rho_{s,t}))Q^2$$

under suitable conditions on  $f$ , cf. Proposition 5.2 and Proposition 5.4. In order to sum these two expressions using Propositions 5.2 and 5.4 we first have to sum the kernel  $R_u$  introduced in Proposition 5.4. The summation formula needed is stated in the following lemma, which has been proved by Mizan Rahman. The proof is presented in Appendix B.

**Lemma 5.5.** (Mizan Rahman) *We have for  $|s|, |t| > 1$ , satisfying  $st \notin \pm q^{-\mathbb{N}}$ ,*

$$\begin{aligned} R_{q^{-2}}(\cos \psi; s, t|q^2) &= \frac{(q^2, q^2, -q^2t^2s^2, -t^{-2}s^{-2}, q\frac{t}{s}e^{i\psi}, q\frac{t}{s}e^{-i\psi}; q^2)_\infty}{(s^{-2}, q^2s^{-2}, -q^2, -1, qste^{i\psi}, qste^{-i\psi}; q^2)_\infty} - \\ &\frac{(q^2t^2, q^3\frac{t}{s}e^{i\psi}, q^3\frac{t}{s}e^{-i\psi}, \frac{q}{st}e^{i\psi}, \frac{q}{st}e^{-i\psi}, q^3ste^{i\psi}, q^3ste^{-i\psi}, -qste^{i\psi}, -qste^{-i\psi}; q^2)}{(-1, -q^2, -q^2, -1, q^2e^{2i\psi}, q^2e^{-2i\psi}, qste^{i\psi}, qste^{-i\psi}, q^2s^{-2}; q^2)_\infty} \\ &\times \frac{(-\frac{q}{st}e^{i\psi}, -\frac{q}{st}e^{-i\psi}; q^2)_\infty}{(q^2s^{-2}, q^4t^2; q^2)_\infty} {}_8W_7(q^2t^2; q^2, q\frac{t}{s}e^{i\psi}, q\frac{t}{s}e^{-i\psi}, qste^{i\psi}, qste^{-i\psi}; q^2, q^2) \end{aligned}$$

and it remains valid for the discrete mass points of the measure in Theorem 5.3, i.e. for  $e^{i\psi} = -stq^{1-2k}$ ,  $k \in \mathbb{Z}$ ,  $|-stq^{1-2k}| > 1$ . Explicitly,

$$R_{q^{-2}}(\mu(-stq^{1-2k}); s, t|q^2) = \frac{(q^2, q^2, -q^2t^2s^2, -t^{-2}s^{-2}, -q^{2-2k}t^2, -q^{2k}s^{-2}; q^2)_\infty}{(s^{-2}, q^2s^{-2}, -q^2, -1, -q^{2-2k}s^2t^2, -q^{2k}; q^2)_\infty}.$$

Rahman's lemma gives the explicit evaluation of the Poisson kernel corresponding to the little  $q$ -Jacobi function in one specific point  $u = q^{-2}$ . Note that the explicit expression for  $R_{q^{-2}}(\mu(-stq^{1-2k}); s, t|q^2)$  follows from the fact that the second term in the general expression vanishes since the factor in front of the  ${}_8W_7$ -series is zero and the  ${}_8W_7$ -series is non-singular for this value.

The condition  $st \notin \pm q^{-\mathbb{N}}$  ensures that the right hand side in Lemma 5.5 does not have simple poles for  $\psi = 0$ , or  $\psi = \pi$ . In the subsequent application of Lemma 5.5 we multiply the result by the weight function as in Proposition 5.4, which cancels the poles.



**Theorem 5.6.** *Let  $f$  be a continuous, compactly supported function on the spectrum of  $\pi_{e^{i\theta}}(\rho_{s,t})$  and assume that  $f(\pi_{e^{i\theta}}(\rho_{s,t})|_{V^\theta(t)})PQ^2$  is of trace class on  $V^\theta(t)$  and that  $f(\pi_{e^{i\theta}}(\rho_{s,t})|_{W^\theta(t)})(1-P)Q^2$  is of trace class on  $W^\theta(t)$ . Let  $|s| \geq |t| > q^{-1}$  and  $s^2t^{\pm 2} \notin q^{2\mathbb{Z}}$ , then*

$$\frac{1}{2\pi} \int_0^{2\pi} \text{Tr}|_{\ell^2(\mathbb{Z})} (f(\pi_{e^{i\theta}}(\rho_{s,t}))Q^2) d\theta = C \int_{\mathbb{R}} f(x) d\nu(x; qs/t, qt/s; -qst|q^2),$$

$$C = \frac{(q^2, q^2, -s^2, -q^2s^{-2}, -t^2, -q^2t^{-2}; q^2)_\infty}{(t^2 - 1)(s^2 - 1)}$$

where the measure is defined in (5.11).

*Remark.* Note that the right hand side is symmetric in  $s$  and  $t$  and invariant under  $(s, t) \mapsto (-s, -t)$ . Since  $\pi_{e^{i\theta}} \circ \psi = \pi_{e^{-i\theta}}$  on  $A_q(SU(1,1))$  and  $\psi(\rho_{s,t}) = \rho_{-t, -s}$  with  $\psi$  defined in Remark 3.3 we see that the left hand side is also invariant under  $(s, t) \mapsto (-t, -s)$ . So the condition  $|s| \geq |t|$  is not essential.

*Proof.* Propositions 5.3 and 5.1 imply that the discrete spectrum of  $f(\pi_{e^{i\theta}}(\rho_{s,t}))|_{W^\theta(t)}$  and the discrete spectrum of  $f(\pi_{e^{i\theta}}(\rho_{s,t}))|_{V^\theta(t)}$  do not overlap, but the continuous spectrum is the same in both cases. We consider the continuous and discrete spectrum separately.

Let us consider the absolutely continuous part on  $[-1, 1]$  first. Using Propositions 5.2 and 5.4 we have to consider

$$(5.15) \quad \frac{(1 - q^2/t^2)}{(t^2 - 1)} \frac{(1 - e^{\pm 2i\psi})}{(1 - \frac{qs}{t}e^{\pm i\psi})(1 - \frac{q}{st}e^{\pm i\psi})} {}_8W_7\left(\frac{q^2}{t^2}; q^2, \frac{qs}{t}e^{\pm i\psi}, \frac{q}{st}e^{\pm i\psi}; q^2, q^2\right) \\ + \frac{(q^2s^{-2}, -1, -q^2; q^2)_\infty^2}{(1 - t^{-2})} \frac{R_{q^{-2}}(\cos \psi; s, t|q^2)(e^{\pm 2i\psi}; q^2)_\infty}{(\frac{qe^{\pm i\psi}}{st}, \frac{qte^{\pm i\psi}}{s}, -qste^{\pm i\psi}, -\frac{qe^{\pm i\psi}}{st}; q^2)_\infty},$$

where we have also used (5.11) and the  $\pm$ -signs means that we have to take two terms, one with  $+$  and one with  $-$ . Now we can use Lemma 5.5 to write  $R_{q^{-2}}$  as a sum of a very-well poised  ${}_8W_7$ -series and an explicit term of infinite  $q$ -shifted factorials. The two very-well poised  ${}_8W_7$ -series can be summed using Bailey's summation formula see [17, (2.11.7)]. In this case we write Bailey's formula as, cf. [31, p. 413],

$$\frac{(1 - ab)(1 - e^{\pm 2i\psi})}{(1 - ae^{\pm i\psi})(1 - be^{\pm i\psi})} {}_8W_7(ab; q, ae^{\pm i\psi}, be^{\pm i\psi}; q, q) \\ - \frac{q}{ab} \frac{(1 - q^2/ab)(1 - e^{\pm 2i\psi})}{(1 - qe^{\pm i\psi}/a)(1 - qe^{\pm i\psi}/b)} {}_8W_7\left(\frac{q^2}{ab}; q, \frac{q}{a}e^{\pm i\psi}, \frac{q}{b}e^{\pm i\psi}; q, q\right) \\ = \frac{(ab, q/ab, aq/b, bq/a, q, q; q)_\infty (e^{\pm 2i\psi}; q)_\infty}{(ae^{\pm i\psi}, be^{\pm i\psi}, qe^{\pm i\psi}/a, qe^{\pm i\psi}/b; q)_\infty}.$$

Using Bailey's summation shows that (5.15) equals

$$\begin{aligned} & \frac{(q^2, q^2, -q^2 t^2 s^2, -t^{-2} s^{-2}, -1, -q^2, e^{\pm 2i\psi}; q^2)_\infty}{(1-s^{-2})(1-t^{-2})(qste^{\pm i\psi}, \frac{q}{st}e^{\pm i\psi}, -qste^{\pm i\psi}, -\frac{q}{st}e^{\pm i\psi}; q^2)_\infty} \\ & - \frac{(q^2 t^2, q^2 t^{-2}, q^2 s^{-2}, q^2 s^2, q^2, q^2, e^{\pm 2i\psi}; q^2)_\infty}{(q\frac{t}{s}e^{\pm i\psi}, qste^{\pm i\psi}, q\frac{s}{t}e^{\pm i\psi}, \frac{q}{st}e^{\pm i\psi}; q^2)_\infty} = \frac{(q^2, q^2, e^{\pm 2i\psi}; q^2)_\infty}{(1-t^{-2})(1-s^{-2})(qste^{\pm i\psi}, \frac{q}{st}e^{\pm i\psi}; q^2)_\infty} \\ & \times \left( \frac{(-q^2 t^2 s^2, -t^{-2} s^{-2}, -1, -q^2; q^2)_\infty}{(-qste^{\pm i\psi}, -\frac{q}{st}e^{\pm i\psi}; q^2)_\infty} - \frac{(t^{-2}, q^2 t^2, q^2 s^2, s^{-2}; q^2)_\infty}{(q\frac{s}{t}e^{\pm i\psi}, q\frac{t}{s}e^{\pm i\psi}; q^2)_\infty} \right). \end{aligned}$$

Use the notation  $\Theta(a) = (a, q^2/a; q^2)_\infty$  for a theta-product in base  $q^2$  and  $S(a, b, c, d) = \Theta(a)\Theta(b)\Theta(c)\Theta(d)$  for the product of four theta products in base  $q^2$ . The following identity for theta-products,

$$(5.16) \quad S(x\lambda, x/\lambda, \mu\nu, \mu/\nu) - S(x\nu, x/\nu, \lambda\mu, \mu/\lambda) = \frac{\mu}{\lambda} S(x\mu, x/\mu, \lambda\nu, \lambda/\nu),$$

see [17, Ex. 2.16], can be used to rewrite the term in parentheses as

$$\begin{aligned} & \frac{S(-1, -t^{-2} s^{-2}, qse^{i\psi}/t, qte^{i\psi}/s) - S(t^{-2}, s^{-2}, -qste^{i\psi}, -qe^{i\psi}/st)}{S(-qste^{i\psi}, -qste^{-i\psi}, qse^{i\psi}/t, qse^{-i\psi}/t)} = \\ & - \frac{q}{st} e^{i\psi} \frac{S(qe^{i\psi}/st, e^{-i\psi}/qst, -s^2, -t^2)}{S(-qste^{i\psi}, -qste^{-i\psi}, qse^{i\psi}/t, qse^{-i\psi}/t)} = \\ & \frac{1}{s^2 t^2} \frac{(qe^{\pm i\psi}/st, qste^{\pm i\psi}, -s^2, -q^2 s^{-2}, -t^2, -q^2 t^{-2}; q^2)_\infty}{(qse^{\pm i\psi}/t, qte^{\pm i\psi}/s, -qe^{\pm i\psi}/st, -qste^{\pm i\psi}; q^2)_\infty} \end{aligned}$$

by taking  $x = 1/st$ ,  $\mu = qe^{i\psi}$ ,  $\lambda = -st$ ,  $\nu = s/t$ . Plugging this in for the term in parentheses we have evaluated (5.15) explicitly as

$$\frac{(q^2, q^2, -s^2, -q^2 s^{-2}, -t^2, -q^2 t^{-2}; q^2)_\infty}{(t^2 - 1)(s^2 - 1)} \frac{(e^{\pm 2i\psi}; q^2)_\infty}{(-\frac{q}{st}e^{\pm i\psi}, -qste^{\pm i\psi}, q\frac{s}{t}e^{\pm i\psi}, q\frac{t}{s}e^{\pm i\psi}; q^2)_\infty}.$$

This proves the statement concerning the absolutely continuous part.

It remains to check the discrete mass points. Since  $|s| \geq |t|$ , we only have an infinite set of discrete mass points from Proposition 5.4 and possibly a finite set of discrete mass points from Proposition 5.2. In case discrete mass points arise from Proposition 5.2 we have to verify

$$\frac{-1}{1-q^2} \frac{w_k(qs/t; q/st, qt/s, qst|q^2)}{h_0(qs/t, q/st, qt/s, qst|q^2)} = C w_k(qs/t; qt/s, -q/st, -qst|q^2)$$

and this is a straightforward calculation using (5.7) and the value for  $C$ . For the infinite set of discrete mass points arising from Proposition 5.4 we have by Lemma 5.5 and (5.12)

$$\begin{aligned} & \frac{(q^2 s^{-2}, -1, -q^2; q^2)_\infty^2}{1-t^{-2}} R_{q^{-2}}(\mu(-stq^{1-2k}); s, t|q^2) \\ & \times \text{Res}_{z=-stq^{1-2k}} \left( z^{-1} w(z; \frac{q}{st}, q\frac{t}{s}, -\frac{q}{st}, -qst|q^2) \right) = \frac{q^{2(k-1)}(s^2 t^2 q^{2-4k} - 1)}{(t^2 - 1)(s^2 - 1)} \end{aligned}$$

using (2.9). From (5.12) and (2.9) we also obtain

$$C \operatorname{Res}_{z=-stq^{1-2k}} z^{-1} w(z; q \frac{s}{t}, q \frac{t}{s}, -\frac{q}{st}, -qst|q^2) = \frac{q^{2(k-1)}(s^2 t^2 q^{2-4k} - 1)}{(t^2 - 1)(s^2 - 1)},$$

so that we have the desired result for the infinite set of discrete mass points.  $\square$

It follows directly from (5.1) and (5.8) that the unitary operator  $T_t(e^{i\theta})$  defined by  $w_p^\theta(t) \mapsto e^{-2ip\theta} w_p^0(t)$  and  $v_p^\theta(t) \mapsto e^{-2ip\theta} v_p^0(t)$  satisfies  $T_t(e^{i\theta}) \pi_{e^{i\theta}}(\rho_{s,t}) T_t(e^{i\theta})^* = \pi_1(\rho_{s,t})$ . Note that  $T_t(e^{i\theta})$  is unitary by Proposition 4.1. So using [15, Ch. II.2, §6] we find that for a bounded continuous function  $f$

$$(5.17) \quad f(\pi(\rho_{s,t})) = \frac{1}{2\pi} \int_0^{2\pi} f(\pi_{e^{i\theta}}(\rho_{s,t})) d\theta = T_t^*(\operatorname{id} \otimes f(\pi_1(\rho_{s,t}))) T_t^*,$$

$T_t = \frac{1}{2\pi} \int_0^{2\pi} T_t(e^{i\theta}) d\theta$ , using  $L^2(\mathbb{T}; \ell^2(\mathbb{Z})) \cong L^2(\mathbb{T}) \otimes \ell^2(\mathbb{Z})$  as tensor product of Hilbert spaces, cf. (4.1). As before,  $T_t$  commutes with multiplication by a function from  $L^2(\mathbb{T})$ , so that  $f(\pi(\rho_{s,t}))$  is decomposable. So we can apply the Haar functional to it, see Remark 2.11.

**Corollary 5.7.** *Let  $|s| \geq |t| > q^{-1}$  and  $s^2 t^{\pm 2} \notin q^{2\mathbb{Z}}$ . Let  $f$  be a continuous, compactly supported function, such that  $f(\pi_1(\rho_{s,t})) Q^2$  is of trace class on  $\ell^2(\mathbb{Z})$ , then  $f(\pi(\rho_{s,t}))$  is a decomposable operator from  $\mathcal{B}(L^2(\mathbb{T}; \ell^2(\mathbb{Z})))$  and*

$$h(f(\pi(\rho_{s,t}))) = \frac{(q^2, q^2, -s^2, -q^2 s^{-2}, -t^2, -q^2 t^{-2}; q^2)_\infty}{(t^2 - 1)(s^2 - 1)} \int_{\mathbb{R}} f(x) d\nu(x; qs/t, qt/s; -qst|q^2)$$

with the measure defined in (5.11).

*Proof.* Since  $f(\pi_1(\rho_{s,t})) Q^2$  is of trace class on  $\ell^2(\mathbb{Z})$ , we have  $f(\pi_1(\rho_{s,t})) P Q^2$  as trace class operator on  $V^0(t)$  and  $f(\pi_1(\rho_{s,t}))(1 - P) Q^2$  as trace class operator on  $W^0(t)$ . Then  $f(\pi_{e^{i\theta}}(\rho_{s,t})) P Q^2$  and  $f(\pi_{e^{i\theta}}(\rho_{s,t}))(1 - P) Q^2$  are trace class operators on  $V^\theta(t)$  and  $W^\theta(t)$ . Now apply Theorem 5.6 and Remark 2.11.  $\square$

**5.4 The Haar functional.** In this subsection we give the measure for the Haar functional on a specific bi- $K$ -type of the Cartan decomposition of Theorem 3.6. This gives an explicit measure space of Askey-Wilson type. The Haar functional on bi- $K$ -invariant elements is obtained in Corollary 5.7.

In order to describe the Haar functional on the non-trivial  $K$ -types of the Cartan decomposition of Theorem 3.6 we need to generalise the measure  $d\nu(\cdot; a, b, d|q)$ . Define, cf. (5.6),

$$W_r(z; a, b, c; d|q) = \frac{(z^2, z^{-2}, qz/d, q/zd; q)_\infty}{(rdz, rd/z, qz/rd, q/rdz, az, a/z, bz, b/z, cz, c/z; q)_\infty}$$

and observe that it differs from the Askey-Wilson weight function by a quotient of theta functions;  $W_r(z; a, b, c; d|q) = \psi_r(z) w(z; a, b, c, d|q)$  with

$$\psi_r(z) = \frac{(dz, q/dz, d/z, qz/d; q)_\infty}{(rdz, q/rdz, rd/z, qz/rd; q)_\infty} = \hat{\psi}_r(\mu(z)).$$

The corresponding measure is defined in terms of the Askey-Wilson measure of (5.5) by

$$(5.18) \quad \int_{\mathbb{R}} f(x) d\nu_r(x; a, b, c; d|q) = h_0(a, b, c, d|q) \int_{\mathbb{R}} f(x) \hat{\psi}_r(x) dm(x; a, b, c, d|q) \\ + \sum_{k \in \mathbb{Z}, |rdq^{-k}| > 1} f(\mu(q^{-k}rd)) \text{Res}_{z=rdq^{-k}} \frac{W_r(z; a, b, c; d|q)}{z},$$

cf. (5.11). Note that any possible discrete mass points of the Askey-Wilson measure at  $\mu(dq^k)$ ,  $k \in \mathbb{Z}_{\geq 0}$ , are annihilated by  $\hat{\psi}_r(\mu(dq^k)) = \psi_r(dq^k) = 0$ . So the support of the measure defined in (5.18) is given by  $[-1, 1]$ , where the measure is absolutely continuous, plus a finite discrete set of points of the form  $\mu(eq^k)$ ,  $k \in \mathbb{Z}_{\geq 0}$  such that  $|eq^k| > 1$  for  $e = a, b$  or  $c$  and an infinite discrete set  $\mu(rdq^k)$ ,  $k \in \mathbb{Z}$ , with  $|rdq^k| > 1$ . Note also that for  $r = -1$ ,  $c = q/d$  we obtain the definition (5.11) of  $d\nu(\cdot; a, b; d|q)$  as a special case. Note that the measure  $d\nu_r(\cdot; a, b, c; d|q)$  is symmetric in  $a, b$ , and  $c$ . The measure  $d\nu_r(\cdot; a, b, c; d|q)$  is positive if  $r < 0$ ,  $0 < b \leq a < d/q$ ,  $0 < c \leq a < d/q$ ,  $bd \geq q$ ,  $cd \geq q$ ,  $ab < 1$ ,  $ac < 1$ , where we assume that  $a$  is the largest of the parameters  $a, b$  and  $c$ . For the general discussion of this measure we refer to [29].

For an element  $\xi^{(i)} = \Gamma_{l,m}^{(i)}(s, t) f(\rho_{s,t})$  corresponding to the Cartan decomposition of Theorem 3.6, we define the corresponding quadratic form

$$\langle \xi^{(i)}, \xi^{(i)} \rangle = h(\bar{f}(\pi(\rho_{s,t})) \pi(\Gamma_{l,m}^{(i)}(s, t) * \Gamma_{l,m}^{(i)}(s, t)) f(\pi(\rho_{s,t}))).$$

By (5.17) and (3.19) we regard the operator in parentheses as a decomposable operator for suitable  $f$  and we assume it satisfies the conditions of Remark 2.11, cf. §4.4.

**Theorem 5.8.** *Let  $f$  be a continuous, compactly supported function on the spectrum of  $\pi_1(\rho_{s,t})$  and  $l \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ ,  $m \in \{-l, -l+1, \dots, l\}$ ,  $|s| \geq |t| > q^{-1}$ ,  $s^2 t^{\pm 2} \notin q^{2\mathbb{Z}}$ . Assume that*

$$\bar{f}(\pi_1(\rho_{s,t})) \pi_1(\Gamma_{l,m}^{(i)}(s, t) * \Gamma_{l,m}^{(i)}(s, t)) f(\pi_1(\rho_{s,t})) Q^2$$

*is of trace class on  $\ell^2(\mathbb{Z})$ . Then we have for positive constants  $C_i$  independent of  $f$ ,*

$$\begin{aligned} \langle \xi^{(1)}, \xi^{(1)} \rangle &= C_1 \int_{\mathbb{R}} |f(x)|^2 d\nu_{-s^2 t^2}(x; \frac{s}{t} q^{1+2l-2m}, q \frac{t}{s}, q^{1+2l+2m} st; \frac{q}{st} |q^2), \\ \langle \xi^{(2)}, \xi^{(2)} \rangle &= C_2 \int_{\mathbb{R}} |f(x)|^2 d\nu_{-1}(x; \frac{s}{t} q^{1+2l+2m}, q \frac{t}{s}, q^{1+2l-2m}/st; qst |q^2), \\ \langle \xi^{(3)}, \xi^{(3)} \rangle &= C_3 \int_{\mathbb{R}} |f(x)|^2 d\nu_{-s^2 t^2}(x; q \frac{s}{t}, \frac{t}{s} q^{1+2l-2m}, q^{1+2l+2m} st; \frac{q}{st} |q^2), \\ \langle \xi^{(4)}, \xi^{(4)} \rangle &= C_4 \int_{\mathbb{R}} |f(x)|^2 d\nu_{-1}(x; q \frac{s}{t}, \frac{t}{s} q^{1+2l+2m}, q^{1+2l-2m}/st; qst |q^2). \end{aligned}$$

*Proof.* This is a direct consequence of Corollary 5.7 and (3.19), cf. the proof of Theorem 4.5.  $\square$

Observe that the quadratic forms in Theorem 5.8 are not always positive definite, cf. Remark 4.6.

The content of Remark 4.7 applies here as well up to some minor changes.

## 6. SPHERICAL FOURIER TRANSFORMS

In this section we give a formal interpretation of the Askey-Wilson function transform as studied in [29] as a Fourier transform on the quantum  $SU(1,1)$  group. Parts of these results only hold at a formal level, so this section mainly serves as the motivation for the study of the Askey-Wilson function transform. In this section we derive which symmetric operators on the function spaces of Theorem 5.8 have to be studied and what are the natural eigenfunctions of this operator to be considered. For the spherical case we calculate the spherical functions and the related action of the Casimir operator. For the Fourier transforms related to the other parts of the Cartan decomposition we only sketch parts of the formal arguments.

**6.1 Unitary representations of  $U_q(\mathfrak{su}(1,1))$ .** The irreducible unitary representations, i.e.  $*$ -representations, of  $U_q(\mathfrak{su}(1,1))$ , are known, see Burban and Klimyk [8], Masuda et al. [41], Vaksman and Korogodskii [57]. We are only interested in the admissible representations, i.e. we require that the eigenvalues of  $A$  are contained in  $q^{\frac{1}{2}\mathbb{Z}}$ , that the corresponding eigenspaces are finite-dimensional, and that the direct sum of the eigenspaces is equal to the representation space. We now recall the classification, see Masuda et al. [41] and Burban and Klimyk [8] for a more general situation. The irreducible admissible unitary representations act in  $\ell^2(\mathbb{Z}_{\geq 0})$  or in  $\ell^2(\mathbb{Z})$ , and we use  $\{e_n\}$  with  $n \in \mathbb{Z}_{\geq 0}$  or  $\mathbb{Z}$  for the standard orthonormal basis of  $\ell^2(\mathbb{Z}_{\geq 0})$  or  $\ell^2(\mathbb{Z})$ . There are, apart from the trivial representation, five types of representations; positive discrete series, negative discrete series, principal unitary series, complementary series and strange series. The representations are in terms of unbounded operators on  $\ell^2(\mathbb{Z})$  or on  $\ell^2(\mathbb{Z}_{\geq 0})$  with common domain the finite linear combinations of the standard basis vectors  $e_k$ , cf. §2.1. We also give the action of the Casimir operator  $\Omega$ , see (3.2), in each of the irreducible admissible representations. The Casimir operator is central, so it acts by a scalar  $[\lambda + \frac{1}{2}]^2$  for some  $\lambda \in \mathbb{C}$ , where  $[a] = (q^a - q^{-a})/(q - q^{-1})$  is the  $q$ -number. The eigenvalues of  $A$  are contained in  $q^{\varepsilon + \mathbb{Z}}$  for  $\varepsilon = 0$  or  $\varepsilon = \frac{1}{2}$ .

In the following list of irreducible admissible unitary representations of  $U_q(\mathfrak{su}(1,1))$  we give the action of the generators of  $U_q(\mathfrak{su}(1,1))$  on the orthonormal basis  $\{e_k\}$ , and the  $\lambda \in \mathbb{C}$  corresponding to the action of  $\Omega$  and  $\varepsilon \in \{0, \frac{1}{2}\}$  corresponding to the set  $q^{\varepsilon + \mathbb{Z}}$  in which  $A$  takes its eigenvalues. We remark that the scalar  $[\lambda + \frac{1}{2}]^2$  and the the eigenvalues of  $A$  determine the irreducible admissible representation of  $U_q(\mathfrak{su}(1,1))$  up to equivalence.

*Positive discrete series.* The representation space is  $\ell^2(\mathbb{Z}_{\geq 0})$ . Let  $k \in \frac{1}{2}\mathbb{N}$ , and  $\lambda = -k$ , and define the action of the generators by

$$\begin{aligned} A \cdot e_n &= q^{k+n} e_n, & D \cdot e_n &= q^{-k-n} e_n, \\ (q^{-1} - q)B \cdot e_n &= q^{-\frac{1}{2}-k-n} \sqrt{(1 - q^{2n+2})(1 - q^{4k+2n})} e_{n+1} \\ (q^{-1} - q)C \cdot e_n &= -q^{\frac{1}{2}-k-n} \sqrt{(1 - q^{2n})(1 - q^{4k+2n-2})} e_{n-1} \end{aligned}$$

with the convention  $e_{-1} = 0$ . We denote this representation by  $T_k^+$ . Now  $\varepsilon = \frac{1}{2}$  if  $k \in \frac{1}{2} + \mathbb{N}$  and  $\varepsilon = 0$  if  $k \in \mathbb{N}$ .

*Negative discrete series.* The representation space is  $\ell^2(\mathbb{Z}_{\geq 0})$ . The negative discrete series representation is  $T_k^- = T_k^+ \circ \phi$ , where  $\phi : U_q(\mathfrak{su}(1, 1)) \rightarrow U_q(\mathfrak{su}(1, 1))$  is the  $*$ -algebra involution defined by  $\phi(A) = D$ ,  $\phi(B) = C$ . The parameters  $\lambda$  and  $\varepsilon$  are the same as for the positive discrete series.

*Principal series.* The representation space is  $\ell^2(\mathbb{Z})$ . Let  $\lambda = -\frac{1}{2} + ib$  with  $0 \leq b \leq -\frac{\pi}{2 \ln q}$  and  $\varepsilon \in \{0, \frac{1}{2}\}$  and assume  $(\lambda, \varepsilon) \neq (-\frac{1}{2}, \frac{1}{2})$ . The action of the generators is defined by

$$\begin{aligned} A \cdot e_n &= q^{n+\varepsilon} e_n, & D \cdot e_n &= q^{-n-\varepsilon} e_n, \\ (q^{-1} - q)B \cdot e_n &= q^{-\frac{1}{2}-n-\varepsilon-ib}(1 - q^{1+2n+2\varepsilon+2ib}) e_{n+1}, \\ (q^{-1} - q)C \cdot e_n &= -q^{\frac{1}{2}-n-\varepsilon+ib}(1 - q^{-1+2n+2\varepsilon-2ib}) e_{n-1}. \end{aligned}$$

We denote the representation by  $T_{\lambda, \varepsilon}^P$ . In case  $(\lambda, \varepsilon) = (-\frac{1}{2}, \frac{1}{2})$  this still defines an admissible unitary representation. It splits as the direct sum  $T_{-\frac{1}{2}, \frac{1}{2}}^P = T_{\frac{1}{2}}^+ \oplus T_{\frac{1}{2}}^-$  of a positive and negative discrete series representation by restricting to the invariant subspaces  $\text{span}\{e_n \mid n \geq 0\}$  and to  $\text{span}\{e_n \mid n < 0\}$ .

*Complementary series.* The representation space is  $\ell^2(\mathbb{Z})$ . Let  $\varepsilon = 0$  and  $-\frac{1}{2} < \lambda < 0$ . The action of the generators is defined by

$$\begin{aligned} A \cdot e_n &= q^n e_n, & D \cdot e_n &= q^{-n} e_n, \\ (q^{-1} - q)B \cdot e_n &= q^{-n-\frac{1}{2}} \sqrt{(1 - q^{2\lambda+2n+2})(1 - q^{2n-2\lambda})} e_{n+1}, \\ (q^{-1} - q)C \cdot e_n &= -q^{-n+\frac{1}{2}} \sqrt{(1 - q^{2\lambda+2n})(1 - q^{2n-2\lambda-2})} e_{n-1}. \end{aligned}$$

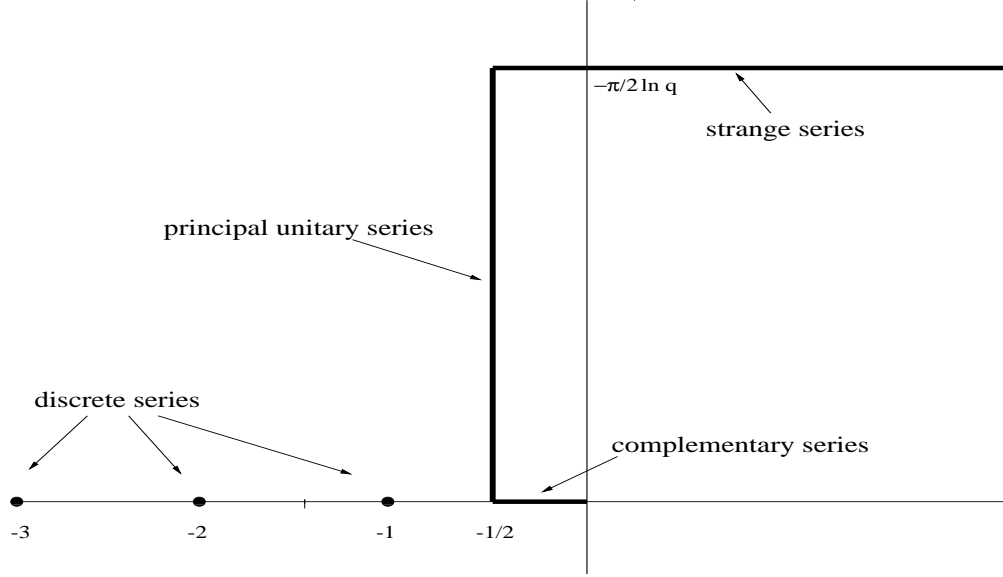
We denote this representation by  $T_{\lambda, 0}^C$ .

*Strange series.* The representation space is  $\ell^2(\mathbb{Z})$ . Let  $\varepsilon \in \{0, \frac{1}{2}\}$ , and put  $\lambda = -\frac{1}{2} - \frac{i\pi}{2 \ln q} + a$ ,  $a > 0$ . The action of the generators is defined by

$$\begin{aligned} A \cdot e_n &= q^{n+\varepsilon} e_n, & D \cdot e_n &= q^{-n-\varepsilon} e_n, \\ (q^{-1} - q)B \cdot e_n &= q^{-n-\varepsilon-\frac{1}{2}} \sqrt{(1 + q^{2n+2\varepsilon+1+2a})(1 + q^{2n+2\varepsilon-2a+1})} e_{n+1}, \\ (q^{-1} - q)C \cdot e_n &= -q^{-n-\varepsilon+\frac{1}{2}} \sqrt{(1 + q^{2n+2\varepsilon-1+2a})(1 + q^{2n+2\varepsilon-2a-1})} e_{n-1}. \end{aligned}$$

We denote this representation by  $T_{\lambda, \varepsilon}^S$ .

*Remark 6.1.* The matrix elements of the irreducible admissible unitary representations in terms of the standard basis  $\{e_k\}$  satisfy (3.15) for  $(s, t) = (\infty, \infty)$ , and can be written in the form  $\Gamma_{l, m}^{(i)}(\infty, \infty)g(\rho_{\infty, \infty})$ , where  $g$  is a power series and  $\rho_{\infty, \infty} = \gamma^* \gamma$ , cf. the Cartan decomposition of Theorem 3.6. The corresponding power series have been calculated explicitly in [41], see also [57], using the explicit duality between  $U_q(\mathfrak{su}(1, 1))$  and  $A_q(SU(1, 1))$ . The power series is a  ${}_2\varphi_1$ -series, and can be interpreted as a little  $q$ -Jacobi

FIGURE 6.1. Representation label  $\lambda$  in the complex plane for  $\varepsilon = 0$ .

function. In the next subsections we compute explicitly matrix coefficients which behave as a character under the left  $AY_t$ , respectively right  $Y_sA$ -action, and we identify them with big  $q$ -Jacobi functions and with Askey-Wilson functions.

**6.2  $K$ -fixed vectors.** In this subsection we look for (generalised) eigenvectors of  $Y_sA \in U_q(\mathfrak{su}(1,1))$  for the eigenvalue zero. For the discrete series representations this involves the Al-Salam and Chihara polynomials. For the other series this involves transforms with a  ${}_2\varphi_1$ -series as kernel, which can be considered as Al-Salam and Chihara functions. For each of the series of representations of  $U_q(\mathfrak{su}(1,1))$  we use the notation of §6.1.

*Irreducible representations in  $\ell^2(\mathbb{Z}_{\geq 0})$ .* For the positive discrete series representation the spectrum of  $Y_sA$  has been calculated in [30, §4] using the Al-Salam and Chihara polynomials, see §5.1. From [30, Prop. 4.1] we conclude that zero is a (generalised) eigenvalue of  $T_k^+(Y_sA)$  if and only if  $\mu(s)$  is in the support of the orthogonality measure  $dm(\cdot; q^{2k}s, q^{2k}/s|q^2)$ , where we use the notation of §5.1. Since  $|s| > 1$  we have  $|\mu(s)| > 1$ , so this can only happen if there exists a discrete mass point. As  $k > 0$  implies  $|q^{2k}/s| < 1$ , we have to have  $q^{2k+2n}s = s$  for some  $n \in \mathbb{Z}_{\geq 0}$ , which is not possible. We conclude that in the positive discrete series we do not have eigenvectors of  $Y_sA$  for the eigenvalue zero.

It follows immediately from §6.1 and (3.6) that for the spectrum of  $T_k^-(Y_sA)$  we have to study the recurrence relation

$$(6.1) \quad \begin{aligned} ((q - q^{-1})T_k^-(Y_sA) - s - s^{-1}) \cdot e_n &= a_n e_{n-1} + b_n e_n + a_{n+1} e_{n+1}, \quad n \in \mathbb{Z}_{\geq 0}, \\ a_n &= q^{1-2k-2n} \sqrt{(1 - q^{2n})(1 - q^{4k+2n-2})}, \quad b_n = -q^{-2k-2n}(s + s^{-1}). \end{aligned}$$

In order to determine the spectrum we have to study the corresponding orthonormal

polynomials  $x p_n(x) = a_n p_{n-1}(x) + b_n p_n(x) + a_{n+1} p_{n+1}(x)$ . We let

$$p_n(x) = (-1)^n q^{n(1+2k)} \frac{(q^2; q^2)_{\frac{1}{2}n}}{(q^{4k}; q^2)_{\frac{1}{2}n}} P_n(x),$$

so that  $P_n(x)$  satisfies the recurrence relation

$$(1 - q^{2n+2})P_{n+1}(x) = (-q^{-2k}(s + s^{-1}) - xq^{2n})P_n(x) - (q^{-4k} - q^{2n-2})P_{n-1}(x).$$

This is precisely the form of the Al-Salam and Chihara polynomials in base  $q^{-2} > 1$  as studied by Askey and Ismail [3, §3.12, §3.13]. It follows from [3, Thm. 3.2] that for  $|s| \geq q^{-1}$  the associated moment problem is determinate, and in that case the support of the orthogonality measure is  $\{2\mu(-sq^{-2p-2k}) \mid p \in \mathbb{Z}_{\geq 0}\}$ , [3, (3.80)-(3.82)]. Part of this statement can also be found in [2, p.26]. So we see that  $T_k^-(Y_s A)$  has an eigenvalue zero if and only if  $2\mu(-s)$  is in the support of the orthogonality measure. Since  $k > 0$  and  $s \in \mathbb{R}$ ,  $|s| \geq q^{-1}$ , we see that this is impossible. We conclude that in the negative discrete series we do not have eigenvectors of  $Y_s A$  for the eigenvalue zero.

We formalise this into the following lemma.

**Lemma 6.2.** *Let  $s \in \mathbb{R}$ . For  $|s| \geq 1$ , the operator  $T_k^+(Y_s A)$  is self-adjoint, and zero is not contained in its spectrum. For  $|s| \geq q^{-1}$  the operator  $T_k^-(Y_s A)$  is essentially self-adjoint, and zero is not contained in its spectrum.*

*Irreducible representations in  $\ell^2(\mathbb{Z})$ .* For the spectrum of  $Y_s A$  in the case of the principal unitary, complementary and strange series representations we have to study the recurrence relation

$$(6.2) \quad ((q^{-1} - q)T_{\lambda, \varepsilon}^\bullet(Y_s A) + s + s^{-1}) \cdot e_n = a_n^\bullet e_{n-1} + b_n e_n + \overline{a_{n+1}^\bullet} e_{n+1},$$

with  $b_n = q^{2n+2\varepsilon}(s + s^{-1})$  and  $\bullet \in \{P, C, S\}$  for the principal unitary, complementary or strange series. The various values for  $a_n^\bullet$ ,  $\bullet \in \{P, C, S\}$ , follow immediately from (3.6) and §6.1;

$$(6.3) \quad \begin{aligned} a_n^P &= q^{ib}(1 - q^{2n-1+2\varepsilon-2ib}), & \varepsilon \in \{0, \frac{1}{2}\}, & \quad 0 \leq b \leq \frac{-\pi}{2 \ln q}, \quad (b, \varepsilon) \neq (0, \frac{1}{2}), \\ a_n^C &= \sqrt{(1 - q^{2n+2\lambda})(1 - q^{2n-2-2\lambda})}, & -\frac{1}{2} < \lambda < 0, \\ a_n^S &= \sqrt{(1 + q^{2n+2\varepsilon-1+2a})(1 + q^{2n+2\varepsilon-1-2a})}, & \varepsilon \in \{0, \frac{1}{2}\}, \quad a > 0. \end{aligned}$$

Define a new orthonormal basis  $\{f_k^\bullet\}_{k \in \mathbb{Z}}$  of  $\ell^2(\mathbb{Z})$  by  $f_n^\bullet = e^{i\phi_n^\bullet} e_{-n}$ , with  $\phi_n^\bullet$  a sequence of real numbers satisfying  $\phi_n^\bullet = \phi_{n+1}^\bullet - \arg(a_{-n}^\bullet)$ , then

$$((q^{-1} - q)T_{\lambda, \varepsilon}^\bullet(Y_s A) + s + s^{-1}) \cdot f_n^\bullet = a_n f_{n+1}^\bullet + b_n f_n^\bullet + a_{n-1} f_{n-1}^\bullet,$$

with  $a_n, b_n$  as in Lemma A.2 in base  $q^2$  and  $c, d, z$  replaced by  $q^2 s^{-2}, q^{2+2\lambda} s^{-1}, q^{-2\varepsilon-2\lambda}$ , where  $\lambda = -\frac{1}{2} + ib$ ,  $0 \leq b \leq -\pi/2 \ln q$ ,  $\varepsilon \in \{0, \frac{1}{2}\}$ ,  $(b, \varepsilon) \neq (0, \frac{1}{2})$  for  $\bullet = P$ ,  $\varepsilon = 0$ ,



$-\frac{1}{2} < \lambda < 0$  for  $\bullet = C$ , and  $\lambda = -\frac{1}{2} - i\pi/2 \ln q + a$ ,  $a > 0$ ,  $\varepsilon \in \{0, \frac{1}{2}\}$  for  $\bullet = S$ . This recurrence relation is related to the second order  $q$ -difference equation for the  ${}_2\varphi_1$ -series, see Appendix A. In case of the principal unitary series, the parameters satisfy the conditions of case (1) of Lemma A.2. For the complementary series, the parameters satisfy the conditions of case (2) of Lemma A.2. and for the strange series, the parameters satisfy the conditions of case (3) of Lemma A.2. Now Theorem A.5 implies the following result, since all conditions are met in the respective cases for  $|s| \geq q^{-1}$ .

**Proposition 6.3.** *Assume  $|s| \geq q^{-1}$  so that  $T_{\lambda,\varepsilon}^\bullet(Y_s A)$  is essentially self-adjoint. With the notation of §6.1 we have that zero is in the discrete spectrum of  $T_{\lambda,0}^\bullet(Y_s A)$ ,  $\bullet \in \{P, C, S\}$ , and zero is not in the spectrum of  $T_{\lambda,\frac{1}{2}}^\bullet(Y_s A)$ . The eigenvector  $v_s^\bullet$  of  $T_{\lambda,0}^\bullet(Y_s A)$  for the eigenvalue zero is given by  $v_s^\bullet =$*

$$(6.4) \quad \sum_{n=-\infty}^{\infty} e^{i(\psi_{-n} + \phi_{-n}^\bullet)} |q^{2-2\lambda} s|^n \sqrt{\frac{(q^{-2\lambda+2n}; q^2)_\infty}{(q^{2\bar{\lambda}+2+2n}; q^2)_\infty}} {}_2\varphi_1 \left( \begin{matrix} q^{2+2\lambda} s^{-2}, q^{2+2\lambda} \\ q^2 s^{-2} \end{matrix}; q^2, q^{-2n-2\lambda} \right) e_n$$

where  $\psi_{k+1} = \psi_k + \arg(q^{2+2\lambda} s^{-1} (1 - q^{-2k+2\bar{\lambda}}))$ . Here we use the analytic continuation of the  ${}_2\varphi_1$ -series as described in §A.2

*Remark.* Since we have  $Y_\infty = A - D$ , we obtain directly from §6.1 that  $T_{\lambda,\varepsilon}^\bullet(A^2 - 1)$  has a vector in its kernel if and only if  $\varepsilon = 0$  and  $\bullet \in \{P, C, S\}$ , and in that case  $e_0$  spans the kernel. We can formally obtain this from Proposition 6.3 by taking termwise limits  $s \rightarrow \infty$ .

**6.3 Zonal spherical functions.** In this subsection we will now give a formal derivation of the zonal spherical functions that may occur in the spherical Fourier transform on the quantum  $SU(1,1)$  group.

With the eigenvectors of Proposition 6.3 at hand we can consider the linear functional

$$(6.5) \quad f_\lambda^\bullet: U_q(\mathfrak{su}(1,1)) \rightarrow \mathbb{C}, \quad X \mapsto \langle T_{\lambda,0}^\bullet(XA) v_t^\bullet, v_s^\bullet \rangle_{\ell^2(\mathbb{Z})},$$

where we take  $s, t \in \mathbb{R}$  with  $|s|, |t| \geq q^{-1}$ . Then we formally have for  $\bullet \in \{P, C, S\}$

$$(6.6) \quad \begin{aligned} (Y_t \cdot f_\lambda^\bullet)(X) &= \langle T_{\lambda,0}^\bullet(XY_t A) v_t^\bullet, v_s^\bullet \rangle_{\ell^2(\mathbb{Z})} = 0, \\ (f_\lambda^\bullet \cdot Y_s)(X) &= \langle T_{\lambda,0}^\bullet(Y_s X A) v_t^\bullet, v_s^\bullet \rangle_{\ell^2(\mathbb{Z})} = \langle T_{\lambda,0}^\bullet(DXA) v_t^\bullet, T_{\lambda,0}^\bullet(Y_s A) v_s^\bullet \rangle_{\ell^2(\mathbb{Z})} = 0 \end{aligned}$$

by Proposition 6.3 and the fact that  $T_{\lambda,0}^\bullet$  is unitary and  $(Y_s A)^* = Y_s A$ . Note that (6.5) and (6.6) can be made rigorous for the limit case  $s \rightarrow \infty$ , so that  $Y_s A$  has to be replaced by  $A^2 - 1$ , which has only an eigenvector for the eigenvalue zero in the principal unitary series, complementary series and the strange series for  $\varepsilon = 0$ . In these cases  $e_0$  is the eigenvector, and the analogue of  $f_\lambda^\bullet: U_q(\mathfrak{su}(1,1)) \rightarrow \mathbb{C}$  for this case is given by  $X \mapsto \langle v_t^\bullet, T_{\lambda,0}^\bullet(AX^*) e_0 \rangle_{\ell^2(\mathbb{Z})}$ , which is well-defined for every  $X \in U_q(\mathfrak{su}(1,1))$  since  $T_{\lambda,0}^\bullet(AX^*) e_0$  has only finitely many terms.

The matrix elements  $T_{\lambda,0;n,m}^\bullet: X \mapsto \langle T_{\lambda,0}^\bullet(X) e_m, e_n \rangle$  have been calculated by Masuda et al. [41], see Remark 6.1. We can formally write

$$(6.7) \quad f_\lambda^\bullet = \sum_{n,m=-\infty}^{\infty} \langle v_t^\bullet, e_m \rangle \langle e_n, v_s^\bullet \rangle q^m T_{\lambda,0;n,m}^\bullet.$$

Note again that for the case  $s \rightarrow \infty$  this can be made rigorous, since the double sum reduces to a single sum and pairing with an arbitrary element  $X \in U_q(\mathfrak{su}(1,1))$  gives a finite sum in this case.

Because of (6.6) we consider  $f_\lambda^\bullet$  as the zonal spherical function. Because of the Cartan decomposition of Theorem 3.6 we formally have that the expression in (6.7) is a function in  $\rho_{s,t}$ ,

$$f_\lambda^\bullet = \sum_{n,m=-\infty}^{\infty} \langle v_t^\bullet, e_m \rangle \langle e_n, v_s^\bullet \rangle q^m T_{\lambda,0;n,m}^\bullet = \phi_\lambda(\rho_{s,t}).$$

In order to determine  $\phi_\lambda$  we evaluate at  $A^\nu$ ,  $\nu \in \mathbb{Z}$ , with  $A^{-1} = D$ . Since  $A$  is group-like, i.e.  $\Delta(A) = A \otimes A$ , we have that pairing with  $A$  is a homomorphism. Since  $\rho_{s,t}(A^\nu) = \mu(q^\nu)$  and  $T_{\lambda,0;n,m}^\bullet(A^\nu) = \langle T_{\lambda,0}^\bullet(A^\nu) e_m, e_n \rangle = q^{n\nu} \delta_{n,m}$  we see that we can determine  $\phi_\lambda$  from

$$\phi_\lambda(\mu(q^\nu)) = \sum_{n=-\infty}^{\infty} \langle v_t^\bullet, e_n \rangle \langle e_n, v_s^\bullet \rangle q^{n(\nu+1)}.$$

Now we can use the following summation formula. This lemma has been proved by Mizan Rahman, and the proof is given in Appendix B.

**Lemma 6.4.** (Mizan Rahman) *Let  $|s|, |t| \geq 1$ , assume  $st > 0$ . For  $\lambda$  corresponding to the principal unitary series, complementary series and strange series, i.e.  $\lambda = -\frac{1}{2} + ib$ ,  $0 \leq b \leq -\pi/2 \ln q$ , or  $-\frac{1}{2} < \lambda < 0$ , or  $\lambda = -\frac{1}{2} + a - i\pi/2 \ln q$ ,  $a > 0$ , and for  $z$  in the annulus  $|q/st| < |z| < |st/q|$  we have*

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \langle v_t^\bullet, e_n \rangle \langle e_n, v_s^\bullet \rangle q^n z^n &= \frac{(q^2, q^2/s^2 t^2, q^{1-2\lambda}/zst, q^{1-2\lambda}z/st; q^2)_\infty}{(q^{-2\lambda}, q^{2-2\lambda}/s^2 t^2, qz/st, q/zst; q^2)_\infty} \\ &\times {}_8W_7(q^{-2\lambda}/s^2 t^2; q^{-2\lambda}/s^2, q^{-2\lambda}/t^2, q^{-2\lambda}, qz/st, q/zst; q^2, q^{2+2\lambda}). \end{aligned}$$

From Lemma 6.4 we formally conclude that the spherical elements of (6.7) can be expressed in terms of a very-well-poised  ${}_8W_7$ -series;

$$(6.8) \quad f_\lambda^\bullet = \phi_\lambda(\rho_{s,t}) = \frac{(q^2, q^2/s^2 t^2, q^{1-2\lambda}/zst, q^{1-2\lambda}z/st; q^2)_\infty}{(q^{-2\lambda}, q^{2-2\lambda}/s^2 t^2, qz/st, q/zst; q^2)_\infty} \times {}_8W_7(q^{-2\lambda}/s^2 t^2; q^{-2\lambda}/s^2, q^{-2\lambda}/t^2, q^{-2\lambda}, qz/st, q/zst; q^2, q^{2+2\lambda}) \Big|_{\mu(z)=\rho_{s,t}}.$$

Note that the right hand side is symmetric in  $z$  and  $z^{-1}$ , so that we can make this specialisation.

*Remark 6.5.* Using the limit transition  $-2\rho_{s,t}/qs \rightarrow \rho_{\infty,t}$  as  $s \rightarrow \infty$ , see (3.12), we formally obtain the limit case  $s \rightarrow \infty$  of the spherical function in (6.8) as

$$\frac{(q^2, -q^{2-2\lambda}\rho_{\infty,t}/t; q^2)_\infty}{(q^{-2\lambda}, -q^2\rho_{\infty,t}/t; q^2)_\infty} {}_3\varphi_2 \left( \begin{matrix} q^{-2\lambda}t^{-2}, q^{-2\lambda}, -q^2\rho_{\infty,t}t^{-1} \\ q^2t^{-2}, -q^{2-2\lambda}\rho_{\infty,t}t^{-1} \end{matrix}; q^2, q^{2+2\lambda} \right),$$

which is, up to a scalar, a special case of the function considered in [28, (3.8)] and is the big  $q$ -Legendre function. Next letting  $t \rightarrow \infty$  using  $t^{-1}\rho_{\infty,t} \rightarrow \rho_{\infty,\infty} = \gamma^*\gamma$ , cf. (3.12), we see that the spherical function in the case  $s, t \rightarrow \infty$  is

$$\begin{aligned} \frac{(q^2, -q^{2-2\lambda}\rho_{\infty,\infty}; q^2)_{\infty}}{(q^{-2\lambda}, -q^2\rho_{\infty,\infty}; q^2)_{\infty}} {}_2\varphi_1 \left( \begin{matrix} q^{-2\lambda}, -q^2\rho_{\infty,\infty} \\ -q^{2-2\lambda}\rho_{\infty,\infty} \end{matrix}; q^2, q^{2+2\lambda} \right) \\ = \frac{(q^2, q^2; q^2)_{\infty}}{(q^{-2\lambda}, q^{2+2\lambda}; q^2)_{\infty}} {}_2\varphi_1 \left( \begin{matrix} q^{2+2\lambda}, q^{-2\lambda} \\ q^2 \end{matrix}; q^2, -q^2\rho_{\infty,\infty} \right) \end{aligned}$$

by [17, (1.4.5)]. This gives back the spherical function, the little  $q$ -Legendre function, as studied by Kakehi, Masuda and Ueno [22] and Vaksman and Korogodskii [57]. So the function  $\phi_{\lambda}$  of (6.8) is a 2-parameter extension of the little  $q$ -Legendre function.

**6.4 The action of the Casimir element.** Since the Casimir element  $\Omega$  acts in any of the irreducible unitary representations of §6.1 by the constant  $[\lambda + \frac{1}{2}]^2$ , we see from (6.7) that we formally have that the spherical function is an eigenfunction of the action of the Casimir operator;  $\Omega.f_{\lambda}^{\bullet} = [\lambda + \frac{1}{2}]^2 f_{\lambda}^{\bullet}$ .

On the other hand, observe that the  $(s, t)$ -spherical elements as defined in Proposition 3.4 are invariant under the action of the Casimir operator, since  $\Omega$  is in the centre of  $U_q(\mathfrak{su}(1, 1))$ . So we can restrict its action to the subalgebra of  $(s, t)$ -spherical elements, or the subalgebra generated by  $\rho_{s,t}$ . For this we have to calculate the radial part of  $\Omega$ , and this is stated in the following lemma. The proof is the same as Koornwinder's proof of [35, Lemma 5.1], so we skip the proof.

**Lemma 6.6.** *Put*

$$\psi(z) = \frac{(1 - qstz)(1 - qsz/t)(1 - qzt/s)(1 - qz/st)}{(1 - z^2)(1 - q^2z^2)},$$

then

$$q(q^{-1} - q)^2 A^{\nu} \Omega \equiv \psi(q^{\nu})(A^{\nu+2} - A^{\nu}) + \psi(q^{-\nu})(A^{\nu-2} - A^{\nu}) + (1 - q)^2 A^{\nu}$$

modulo  $U_q(\mathfrak{su}(1, 1))Y_t + Y_s U_q(\mathfrak{su}(1, 1))$ .

As for the polynomial case discussed by Koornwinder [35], we derive from this equation that the action of the Casimir operator on the subalgebra of  $(s, t)$ -spherical elements is given by the Askey-Wilson  $q$ -difference operator [5];

$$(6.9) \quad q(q^{-1} - q)^2 \Omega.(f(\rho_{s,t})) = \left( \psi(q^{\nu})(f(\mu(q^{\nu+2})) - f(\mu(q^{\nu}))) + \right. \\ \left. \psi(q^{-\nu})(f(\mu(q^{\nu-2})) - f(\mu(q^{\nu}))) + (1 - q)^2 f(\mu(q^{\nu})) \right) \Big|_{\mu(q^{\nu}) = \rho_{s,t}}.$$

Combining (6.9) with the scalar action of  $\Omega$  in the irreducible representations we formally find that the spherical function  $\phi_{\lambda}(\mu(z))$  is an eigenfunction of

$$(6.10) \quad \begin{aligned} L\phi_{\lambda}(\mu(z)) &= (-1 - q^2 + q(q^{2\lambda+1} + q^{-2\lambda-1}))\phi_{\lambda}(\mu(z)), \\ L &= \psi(z)(T_{q^2} - 1) + \psi(z^{-1})(T_{q^{-2}} - 1), \quad (T_q f)(z) = f(qz). \end{aligned}$$

This is only a formal derivation, due to the fact that the series (6.7) is only a formal expression. Note that the eigenvalues in (6.10) are real for  $\lambda$  corresponding to the principal unitary series, complementary series and strange series. The function  $\phi_\lambda(\mu(z))$  given in (6.8) is indeed an eigenfunction of the Askey-Wilson  $q$ -difference equation as in (6.10), see Ismail and Rahman [19], Suslov [55], [56]. So we call  $\phi_\lambda$  of (6.8) an Askey-Wilson function.

*Remark 6.7.* For the limit case  $s \rightarrow \infty$  we obtain the same eigenvalue equation as in (6.10) but now with the operator

$$(6.11) \quad \begin{aligned} L &= A(z)(T_{q^2} - 1) + B(z)(T_{q^{-2}} - 1), \\ A(z) &= q^2(1 + \frac{1}{q^2tz})(1 + \frac{t}{q^2z}), \quad B(z) = (1 + \frac{1}{tz})(1 + \frac{t}{z}). \end{aligned}$$

Then it is known [18] that the spherical function given in Remark 6.5 is indeed a solution to the eigenvalue equation. See [28] for more information. For the limit case  $s, t \rightarrow \infty$  we find the same eigenvalue equation (6.11), but with now  $A(z) = q^2(1 + q^{-2}z^{-1})$  and  $B(z) = 1 + z^{-1}$ . The little  $q$ -Legendre function as in Remark 6.5 is a solution of the eigenvalue equation as follows from (A.8), see also [21], [22], [57].

**Proposition 6.8.** *The action of the Casimir operator on the space of  $(s, t)$ -spherical elements is symmetric, i.e.*

$$\int_{\mathbb{R}} (Lf)(x)\bar{g}(x) d\nu(x; qs/t, qt/s; -qst|q^2) = \int_{\mathbb{R}} f(x)\overline{(Lg)(x)} d\nu(x; qs/t, qt/s; -qst|q^2)$$

for continuous, compactly supported functions  $f$  and  $g$  such that the functions  $F(z) = f(\mu(z))$  and  $G(z) = g(\mu(z))$  have an analytic continuation to a neighbourhood of  $\{z \in \mathbb{C} \mid q^2 \leq |z| \leq q^{-2}\}$ .

So we interpret this as  $h(g(\rho_{s,t})^*\Omega.f(\rho_{s,t})) = h((\Omega.g(\rho_{s,t}))^*f(\rho_{s,t}))$  using Corollary 5.7.

*Proof.* This is a calculation using Cauchy's theorem and shifting sums, see [29] for details.  $\square$

Proposition 6.8 remains valid for the limit case  $s \rightarrow \infty$  with the same proof, see [28]. Taking furthermore  $t \rightarrow \infty$  leads to the situation considered by Takechi, Masuda and Ueno [22], see also [21], [57].

**6.5 The spherical Fourier transform.** Suppose that  $s \geq t \geq 1$ , and define

$$(6.12) \quad \begin{aligned} \Phi_{\mu(q^{1+2\lambda})}(\mu(x)) &= \frac{(q^{-2\lambda}, q^{2+2\lambda}; q^2)_\infty}{(q^2, q^2, q^{-2\lambda}s^{-2}, q^{2+2\lambda}s^{-2}, q^2t^{-2}; q^2)_\infty} \phi_\lambda(\mu(x)) = \\ &\quad \frac{(q^{3+2\lambda}x^{\pm 1}/st; q^2)_\infty}{(q^{4+2\lambda}t^{-2}, q^{2+2\lambda}s^{-2}, q^2s^{-2}, qx^{\pm 1}/st; q^2)_\infty} \\ &\quad \times {}_8W_7(q^{2+2\lambda}/t^2; q\frac{s}{t}x^{\pm 1}, q^{2+2\lambda}, q^{2+2\lambda}, q^{2+2\lambda}/t^2; q^2, q^{-2\lambda}s^{-2}) \end{aligned}$$

by an application of [17, (III.24)]. Here  $\phi_\lambda$  is defined in (6.8). For (6.12) to be well-defined we need that  $\phi_\lambda$  is invariant under interchanging  $q^{1+2\lambda}$  and  $q^{-1-2\lambda}$ , or changing  $\lambda$  into  $-1-\lambda$ . This is not obvious from (6.8), but it can be obtained from Bailey's transformation for a very-well-poised  ${}_8\varphi_7$ -series [17, (2.10.1)], or directly from the proof of Lemma 6.4 as given in Appendix B. The quantum group theoretic interpretation of the invariance is that the principal unitary, complementary and strange series representations are all obtained from the so-called principal series representations which are equivalent for  $\lambda$  and  $-1-\lambda$ , see Burban and Klimyk [8], Masuda et al. [41].

We now define the spherical Fourier transform of a  $(s, t)$ -spherical element  $\xi = f(\rho_{s,t})$ , with  $f$  continuous and compactly supported on the spectrum of  $\pi_1(\rho_{s,t})$ , by

$$(6.13) \quad (\mathcal{F}\xi)(\sigma) = \int_{\mathbb{R}} f(x) \overline{\Phi_\sigma(x)} d\nu(x; q \frac{t}{s}, q \frac{s}{t}; -qst|q^2) = (\mathcal{F}f)(\sigma),$$

which is, up to constant, formally equal to  $h((\phi_\lambda(\rho_{s,t}))^* \xi)$  with  $\sigma = \mu(q^{1+2\lambda})$ . The spherical Fourier transform (6.13) is a special case of the Askey-Wilson function transform as studied in [29]. There an inversion formula is obtained, which reduces to the following theorem in this situation.

**Theorem 6.9.** *Assume  $s \geq t \geq 1$ . The spherical Fourier transform of (6.13) is inverted by*

$$f(x) = C \int_{\mathbb{R}} (\mathcal{F}f)(z) \Phi_z(x) d\nu_{-q^{-4}s^{-2}}(z; q, q, qt^{-2}; qs^2|q^2),$$

$$C = (qs)^{-1} (q^2, q^2, q^2 t^{-2}, q^2 t^{-2}, q^2 s^{-2}; q^2)_\infty \Theta(-q^2)^2 \Theta(-t^{-2}) \Theta(-s^{-2}),$$

as an identity in  $L^2(\mathbb{R}, d\nu(\cdot; qs/t, qt/s; -qst|q^2))$ . The notation  $\Theta(a) = (a, q^2/a; q^2)_\infty$  for  $a$  (normalised) theta product is used.

*Remark 6.10.* (i) We refer to [29] for complete proofs and the appropriate generalisation of Theorem 6.9. Note that the spherical Fourier transform is self-dual for the case  $s = t = 1$ . In compliance with the situation for the compact quantum  $SU(2)$  group case, we could call the spherical functions for the case  $s = t = 1$  the continuous  $q$ -Legendre functions, cf. [5], [25], [26], [27]. For the  $SU(1,1)$  group the spherical Fourier transform is given by the Legendre function transform, which is also known as the Mehler-Fock transform, see [32], [60, Ch. VI], [61, Ch. 7]. So the transform (6.12) and its inverse of Theorem 6.9 is a two-parameter  $q$ -analogue of the Legendre function (or Mehler-Fock) transform.

(ii) We see that the support of the Plancherel measure of the spherical Fourier transform is  $[-1, 1]$ , which corresponds to all of the principal unitary series representations, plus the discrete set  $\{\mu(-q^{1-2k}) | k \in \mathbb{N}\}$ , which corresponds to the strange series representations with  $\lambda = -1 + k - i\pi/2 \ln q$ ,  $k \in \mathbb{N}$ , (and  $\varepsilon = 0$ ). Note that the support is independent of  $s$  and  $t$ . Indeed, the existence of a non-trivial kernel of  $Y_s A$  in an admissible irreducible unitary representation of  $U_q(\mathfrak{su}(1,1))$  is independent of  $s$ .

(iii) For the limiting cases we obtain the big  $q$ -Legendre function transform, which is studied and inverted in [28], for  $s \rightarrow \infty$ , and the little  $q$ -Legendre function transform, which is studied and inverted in [22], [57], [21], Appendix A, for  $s, t \rightarrow \infty$ . In all these cases the support of the Plancherel measure is as in part (ii) of this Remark.

**6.6 Other  $K$ -types.** In the previous subsections we have interpreted in a formal way a special (2 continuous parameters) case of the Askey-Wilson function transform as the spherical Fourier transform on the quantum  $SU(1, 1)$  group. This is connected to the  $(s, t)$ -spherical part of the Cartan decomposition in Theorem 3.6. It is also possible to associate a Fourier transform related to the non-trivial  $K$ -types in the Cartan decomposition of Theorem 3.6, and this allows us to interpret a 4-parameter (2 continuous, 2 discrete) case of the Askey-Wilson function transform on the quantum  $SU(1, 1)$  group. Since the derivation lives on the same formal level we only shortly discuss this more general case, and we refer to [29] for the precise analytic proof of the Askey-Wilson function transform. For the limiting cases we refer to [28] for the big  $q$ -Jacobi function transform and to [21], or Appendix A, for the little  $q$ -Jacobi function transform. We stress that the formal results of this subsection have served as the motivation for the analytic definition of the general Askey-Wilson, respectively big  $q$ -Jacobi, function transform in [29], respectively [28].

First we consider the action of the Casimir operator  $\Omega$ . Since  $\Omega$  is central, it preserves the Cartan decomposition. So we have, cf. Theorem 3.6,

$$(6.14) \quad \Omega.(\Gamma_{i,j}^{(p)}(s, t) f(\rho_{s,t})) = \Gamma_{i,j}^{(p)}(s, t) (Lf)(\rho_{s,t})$$

for some linear operator  $L$ . In the rest of this subsection we take  $p = 2$ , the other cases can be treated similarly. In order to determine  $L$  we proceed by determining  $A^\nu \Omega$  modulo  $U_q(\mathfrak{su}(1, 1))(AY_t - \lambda_{-i}(t)) + (Y_s A - \lambda_j(s))U_q(\mathfrak{su}(1, 1))$  with  $\lambda_j(t)$  as in Lemma 3.2. This is done as in Lemma 6.6 using Koornwinder's method, and we find that it is a linear combination of  $A^{\nu+2}$ ,  $A^\nu$  and  $A^{\nu-2}$  with explicit rational coefficients in  $q^\nu$ . Next we evaluate (6.14) in  $A^\nu$ . Since  $A^\nu$  is group-like in  $U_q(\mathfrak{su}(1, 1))$ , i.e.  $\Delta(A^\nu) = A^\nu \otimes A^\nu$ , this is a homomorphism. So

$$\begin{aligned} \Gamma_{i,j}^{(2)}(s, t)(A^\nu) (Lf)(\mu(q^\nu)) &= (\Gamma_{i,j}^{(2)}(s, t) (Lf)(\rho_{s,t}))(A^\nu) = (\Gamma_{i,j}^{(2)}(s, t) f(\rho_{s,t}))(A^\nu \Omega) = \\ &= \psi^+(q^k) \Gamma_{i,j}^{(2)}(s, t)(A^{\nu+2}) f(\mu(q^{\nu+2})) + \psi^0(q^k) \Gamma_{i,j}^{(2)}(s, t)(A^\nu) f(\mu(q^\nu)) \\ &\quad + \psi^-(q^k) \Gamma_{i,j}^{(2)}(s, t)(A^{\nu-2}) f(\mu(q^{\nu-2})) \end{aligned}$$

for certain explicit rational functions  $\psi^+, \psi^0, \psi^-$ . Using (3.16) and the homomorphism property we can calculate  $\Gamma_{i,j}^{(2)}(s, t)(A^\nu)$  explicitly in terms of finite  $q$ -shifted factorials. For  $j \in \{-i, 1-i, \dots, i\}$  we have  $\Gamma_{i,j}^{(2)}(s, t)(A^\nu) = Cq^{-\nu i}(q^{1+\nu}/st; q^2)_{i-j}(q^{1+\nu}s/t; q^2)_{i+j}$  for a non-zero constant  $C$  independent of  $\nu$ . In this way we can determine  $L$  in terms of an Askey-Wilson difference operator. We find

$$(6.15) \quad \begin{aligned} q^{2i+1}(q - q^{-1})^2 L &= \psi(z)(T_{q^2} - 1) + \psi(z^{-1})(T_{q^{-2}} - 1) + (1 - q^{2i+1})^2, \\ \psi(z) &= \frac{(1 - qtz/s)(1 - q^{1+2i-2j}z/st)(1 - qstz)(1 - q^{1+2i+2j}sz/t)}{(1 - z^2)(1 - q^2z^2)}. \end{aligned}$$

With respect to the measure  $d\nu_{-1}(\cdot; q^{1+2i+2j}s/t, qt/s, q^{1+2i-2j}/st; qst|q^2)$ , the operator  $L$  is formally symmetric, cf. Theorem 5.8, Proposition 6.8, and see [29] for the general case. See also [26, §7] for the compact case.

To find the appropriate eigenfunctions of  $L$  we have to determine to which of the irreducible admissible unitary representations of  $U_q(\mathfrak{su}(1,1))$  as in §6.1 we formally can associate an element in the corresponding part of the Cartan decomposition. So we have to determine for which of the representations there exists eigenvectors of  $Y_s A$  and  $Y_t A$  for the eigenvalues  $\lambda_i(s)$  and  $\lambda_j(t)$  as defined in Lemma 3.2 with  $i, j \in \frac{1}{2}\mathbb{Z}$ . This is done in Appendix A, cf. §6.2, where essentially the complete spectral analysis of  $Y_s A$  in any of the irreducible admissible unitary representations is described. Now for the principal unitary, complementary and strange series we have an eigenvector of  $T_{\lambda,\varepsilon}^\bullet(Y_s A)$  for the eigenvalue  $\lambda_i(s)$  for every  $\bullet \in \{P, C, S\}$  and  $\lambda$  with  $\varepsilon \equiv i \pmod{\mathbb{Z}}$ . In order to have  $\lambda_i(s)$  in the discrete spectrum of  $T_{\lambda,\varepsilon}^\bullet(Y_s A)$  we need  $|sq^{2i}| > 1$ . In the discrete series,  $T_k^\pm(Y_s A)$  has an eigenvector for the eigenvalue  $\lambda_i(s)$  for only finitely many values of  $k$ . Moreover, we need  $k \equiv i \pmod{\mathbb{Z}}$  and for  $i < 0$  the eigenvalue can occur only in the negative discrete series and for  $i > 0$  it can occur only in the positive discrete series. Here we assume  $|s| \geq q^{-1}$  so that we are dealing with essentially self-adjoint operators. Let us denote such an eigenvector, if it exists, by  $v_s(i)$ . Assuming that the irreducible admissible unitary representation  $T_{\lambda,\varepsilon}^\bullet$  or  $T_k^\pm$  contains both  $v_s(j)$  and  $v_t(-i)$  we formally see that  $f_\lambda(X) = \langle T_{\lambda,\varepsilon}^\bullet(XA)v_t(-i), v_s(j) \rangle$  satisfies (3.15) with  $\lambda = \lambda_{-i}(t)$ ,  $\mu = \lambda_j(s)$ . In case  $j \in \{-i, 1-i, \dots, i\}$  we formally obtain

$$f_\lambda = \sum_{n,m} \langle v_t(-i), e_m \rangle \langle e_n, v_s(j) \rangle q^{m+\varepsilon} T_{\lambda,\varepsilon;n,m}^\bullet = \Gamma_{i,j}^{(2)}(s,t) \phi_{\lambda,\varepsilon}^{(i,j)}(\rho_{s,t})$$

for some function  $\phi_{\lambda,\varepsilon}^{(i,j)}$ . Here  $n, m$  run through  $\mathbb{Z}$  if  $T_{\lambda,\varepsilon}^\bullet$  is in the principal unitary, complementary or strange series representations and through  $\mathbb{Z}_{\geq 0}$  if  $T_{\lambda,\varepsilon}^\bullet = T_k^\pm$  is in the discrete series representation. Evaluating in  $A^\nu$  we can obtain  $\phi_{\lambda,\varepsilon}^{(i,j)}$  from

$$\sum_n \langle v_t(-i), e_n \rangle \langle e_n, v_s(j) \rangle q^{(n+\varepsilon)(1+\nu)} = C q^{-\nu i} (q^{1+\nu}/st; q^2)_{i-j} (q^{1+\nu}s/t; q^2)_{i+j} \phi_{\lambda,\varepsilon}^{(i,j)}(\mu(q^\nu)).$$

From this we can, in a similar way as for Lemma 6.4 determine  $\phi_{\lambda,\varepsilon}^{(i,j)}$  explicitly for the principal unitary, complementary and strange series representations for the diagonal case  $i = -j$ . An extension of Rahman's method in Appendix B can be used to sum the other cases. (This is pointed out to us by Hjalmar Rosengren.) In case of the positive discrete series the sum runs through  $\mathbb{Z}_{\geq 0}$  and the coefficients of  $v_t(j)$  are Al-Salam and Chihara polynomials, see (5.3) and [30]. The sum can then be evaluated using the Poisson kernel for the Al-Salam and Chihara polynomials obtained by Askey, Rahman and Suslov [4, (14.8)]. In the case  $j \in \{-i, 1-i, \dots, i\}$  we find, up to a scalar independent of  $z$ , in the case of the positive discrete series  $T_k^+$ ,

$$\begin{aligned} \phi_{\lambda,\varepsilon}^{(i,j)}(\mu(z)) &= z^{i-\lambda} \frac{(q^2 z^2, q^{1+2j-2\lambda} stz, q^{1+2j-2\lambda} sz/t, q^{1-2i-2\lambda} stz, q^{1+2i-2\lambda} sz/t; q^2)_\infty}{(q^{2-2\lambda+2j} s^2 z^2, q^{1+2j-2\lambda} stz, qsz/t, qz/st, q^{1-2i-2j} tz/s; q^2)_\infty} \\ &\times {}_8W_7(q^{2j-2\lambda} s^2 z^2; qstz, qsz/t, q^{2j-2\lambda} s^2, q^{1+2j-2i} stz, q^{1+2i+2j} sz/t; q^2, q^{-2\lambda-2j} s^{-2}), \end{aligned}$$

where  $\lambda$  is equal to  $-k$ , see §6.1. After application of [17, (2.10.1)] we can relate the right hand side with the asymptotically free solution of  $Lf(z) = [\frac{1}{2} + \lambda]^2 f(z)$  for  $z \rightarrow 0$  as

considered in [29]. Using the connection coefficient formula [17, (2.11.1)], see [29], we can show that the right hand side is indeed invariant under  $z$  to  $z^{-1}$ , and that it coincides, up to a constant, with the Askey-Wilson function, i.e. the spherical function for the Askey-Wilson function transform, since one of the connection coefficients vanishes. By comparing with [19], [55], [56], we see that these functions are indeed solutions to the Askey-Wilson difference operator of (6.15).

Next we formally associate the corresponding Fourier transform to the diagonal case by

$$f \mapsto \hat{f}(\mu(q^{1+2\lambda})) = h\left(\left(\Gamma_{i,-i}^{(2)}(s, t)\phi_{\lambda, \varepsilon}^{(i, -i)}\right)^* \Gamma_{i,-i}^{(2)}(s, t)f(\rho_{s, t})\right).$$

The explicit expression of  $h$ , see Theorem 5.8, and of  $\phi_{\lambda, \varepsilon}^{(i, -i)}$  can be used to formally invert this transform by a spectral analysis of the operator  $L$  related to the Casimir element  $\Omega$ , assuming that the measure for the case  $p = 2$  in Theorem 5.8 is positive. See [29] for the rigorous analytic derivations and for the explicit inversion formulas. From this result we see that the Plancherel formula is supported on the principal unitary series and the same discrete subset of the strange series, cf. Remark 6.10(ii), plus on the discrete series representations that allow a map  $f_\lambda$  as before, i.e. for those discrete series representations that contain the appropriate eigenvectors of  $Y_s A$  and  $Y_t A$ . For all other cases we can proceed in a similar fashion.

The limit case  $s \rightarrow \infty$  gives a 3-parameter family of big  $q$ -Jacobi function transforms in this way, see [28] for the analytic proofs. Taking moreover  $t \rightarrow \infty$  brings us back to the case studied by Takehi [21].

## APPENDIX A. SPECTRAL ANALYSIS OF A DOUBLY INFINITE JACOBI MATRIX

In this subsection we give the spectral analysis of a doubly infinite Jacobi matrix that arises from the second order  $q$ -difference equation for the basic hypergeometric series  ${}_2\varphi_1$ . In a way the results can be viewed as the spectral analysis of a  $q$ -integral operator on  $(0, \infty)$  with a basic hypergeometric series as kernel. The result covers in particular the little  $q$ -Jacobi function transform as studied by Takehi [21], see also [22], [57]. The method of proof is similar to the one used in [21], so we are brief. The result is more general.

**A.1 Generalities.** In this subsection we collect some generalities on the study of the symmetric operator on the Hilbert space  $\ell^2(\mathbb{Z})$  defined by

$$(A.1) \quad L e_k = a_k e_{k+1} + b_k e_k + \overline{a_{k-1}} e_{k-1}, \quad a_k \neq 0, b_k \in \mathbb{R},$$

where  $\{e_k\}_{k \in \mathbb{Z}}$  is the standard orthonormal basis of  $\ell^2(\mathbb{Z})$ . By replacing  $e_k$  by  $e^{i\psi_k} e_k$  with  $\psi_k = \psi_{k+1} - \arg a_k$  we see that we may assume that  $a_k > 0$ , which we assume in this subsection from now on. We use the standard terminology and results as in Dunford and Schwartz [16, Ch. XII], see also Berezanskiĭ [7], Takehi [21], Takehi et al. [22], Koelink and Stokman [28], Masson and Repka [40], Rudin [51], Simon [54].

The domain  $\mathcal{D}$  of  $L$  is the dense subspace  $\mathcal{D}(\mathbb{Z})$  of finite linear combinations of the basis elements  $e_k$ , then  $L$  is a densely defined symmetric operator. Let  $L^*$  with domain  $\mathcal{D}^*$  be the adjoint and  $L^{**}$  with domain  $\mathcal{D}^{**}$  the closure of  $L$ . The deficiency indices are



equal since  $L$  commutes with complex conjugation and they are less than or equal to 2, so that  $L$  has self-adjoint extensions.

For any two vectors  $u = \sum_{k=-\infty}^{\infty} u(k)e_k$  and  $v = \sum_{k=-\infty}^{\infty} v(k)e_k$  we define the Wronskian by

$$(A.2) \quad [u, v](k) = a_k(u(k+1)v(k) - v(k+1)u(k)).$$

Note that the Wronskian  $[u, v](k)$  is independent of  $k$  if  $Lu = xu$  and  $Lv = xv$  for  $x \in \mathbb{C}$ . In this case we have that  $u$  and  $v$  are linearly independent solutions if and only if  $[u, v] \neq 0$ .

Associated with the operator  $L$  we have two Jacobi matrices  $J^+$  and  $J^-$  acting on  $\ell^2(\mathbb{Z}_{\geq 0})$  with orthonormal basis  $\{f_k\}_{k \in \mathbb{Z}_{\geq 0}}$ , which are given by

$$(A.3) \quad \begin{aligned} J^+ f_k &= \begin{cases} a_k f_{k+1} + b_k f_k + a_{k-1} f_{k-1}, & \text{for } k \geq 1, \\ a_0 f_1 + b_0 f_0, & \text{for } k = 0, \end{cases} \\ J^- f_k &= \begin{cases} a_{-k-1} f_{k+1} + b_{-k} f_k + a_{-k} f_{k-1}, & \text{for } k \geq 1, \\ a_{-1} f_1 + b_0 f_0, & \text{for } k = 0, \end{cases} \end{aligned}$$

initially defined on  $\mathcal{D}(\mathbb{Z}_{\geq 0})$ . Then  $J^{\pm}$  are densely defined symmetric operators with deficiency indices  $(0,0)$  or  $(1,1)$  corresponding to whether the associated Hamburger moment problems is determinate or indeterminate, see Akhiezer [1], Berezanskiĭ [7], Simon [54]. Moreover, by [40, Thm. 2.1] the deficiency indices of  $L$  are obtained by summing the deficiency indices of  $J^+$  and  $J^-$ .

From now on we assume that  $a_k$  is bounded as  $k \rightarrow -\infty$ . Then  $\lim_{m \rightarrow -\infty} [u, \bar{v}](m) = 0$  for  $u, v \in \mathcal{D}^*$ . By [7, Thm. 1.3, p. 504] it follows that  $J^-$  is self-adjoint, hence the space  $S_x^- = \{u \mid Lu = xu, \sum_{k=-\infty}^N |u(k)|^2 < \infty \text{ for some } N \in \mathbb{Z}\}$  is one-dimensional for  $x \in \mathbb{C} \setminus \mathbb{R}$  by [1, §1.3]. Let us say that  $\Phi_x$  spans  $S_x^-$  for  $x \in \mathbb{C} \setminus \mathbb{R}$ . The similarly defined space  $S_x^+ = \{u \mid Lu = xu, \sum_{k=N}^{\infty} |u(k)|^2 < \infty \text{ for some } N \in \mathbb{Z}\}$  is either one-dimensional or two-dimensional according to whether  $J^+$  has deficiency indices  $(0,0)$  or  $(1,1)$ , see [1, Ch. 1].

For the purposes of this appendix, it suffices to consider the case that  $J^+$  has deficiency indices  $(0,0)$ , which we will assume from now on. In particular  $L$  is essentially self-adjoint, i.e.  $\mathcal{D}^* = \mathcal{D}^{**}$ . The closure  $L^{**}$  of  $L$  satisfies the same formula (A.1), so we denote it also by  $L$ . We thus have that  $S_x^+$  is one-dimensional, say spanned by  $\phi_x$ , and we have that  $[\phi_x, \Phi_x] \neq 0$ . Indeed,  $\Phi_x$  cannot be in  $S_x^+$  since otherwise  $Lu = iu$  would have a non-trivial solution in  $\ell^2(\mathbb{Z})$ .

We also have to deal with possible non-real solutions of (A.1). Note that if  $\psi_x$  is a solution of  $L\psi_x = x\psi_x$ , then so is  $\overline{\psi_x}$  defined by  $\overline{\psi_x}(k) = \overline{\psi_x(k)}$ , since we assume that the coefficients  $a_k$  and  $b_k$  are real. Observe in particular, that  $\overline{\phi_x}$  and  $\overline{\Phi_x}$  are multiples of  $\phi_x$  and  $\Phi_x$ , respectively, since the subspaces  $S_x^{\pm}$  are one-dimensional.

Having the solutions  $\Phi_x$  and  $\phi_x$  at hand we can define the Green kernel for  $x \in \mathbb{C} \setminus \mathbb{R}$  by

$$(A.4) \quad G_x(k, l) = \frac{1}{[\Phi_x, \overline{\phi_x}]} \begin{cases} \Phi_x(k) \overline{\phi_x(l)}, & \text{for } k \leq l, \\ \Phi_x(l) \overline{\phi_x(k)}, & \text{for } k \geq l \end{cases}$$

and the operator  $(G_x u)(k) = \langle u, \overline{G_x(k, \cdot)} \rangle = \sum_{l=-\infty}^{\infty} u(l) G_x(k, l)$  for  $u \in \ell^2(\mathbb{Z})$ . Note that this is well defined, since  $G_x(k, \cdot) \in \ell^2(\mathbb{Z})$  for all  $k$ . Special cases of the following proposition are proved in [22], [21] and [11].

**Proposition A.1.** *Let  $L$  with domain  $\mathcal{D}$  be essentially self-adjoint, then the resolvent of the closure of  $L$  is given by  $((x - L)^{-1}u)(k) = \sum_{l=-\infty}^{\infty} u(l) G_x(k, l)$ .*

Since  $L$  with domain  $\mathcal{D}^{**}$  is self-adjoint we have the spectral decomposition,  $L = \int_{\mathbb{R}} t dE(t)$ , for a unique projection valued measure  $E$  on  $\mathbb{R}$ . This means that for any vectors  $u \in \mathcal{D}^{**}$ ,  $v \in \ell^2(\mathbb{Z})$  we have a complex measure  $E_{u,v}$  on  $\mathbb{R}$  such that  $\langle Lu, v \rangle = \int_{\mathbb{R}} t dE_{u,v}(t)$ , where  $E_{u,v}(B) = \langle E(B)u, v \rangle$  for any Borel subset  $B \subset \mathbb{R}$ , see [51, Thm. 13.30]. The measure can be obtained from the resolvent by the inversion formula, see [16, Thm. XII.2.10],

$$(A.5) \quad E_{u,v}((x_1, x_2)) = \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{x_1+\delta}^{x_2-\delta} \langle (x - i\varepsilon - L)^{-1}u, v \rangle - \langle (x + i\varepsilon - L)^{-1}u, v \rangle dx,$$

where  $x_1 < x_2$ . Combined with Proposition A.1, we see that the Wronskian is crucial for the structure of the spectral measure. In particular, if  $\langle (x - L)^{-1}u, v \rangle$  is meromorphic in a subset of  $\mathbb{C}$  we find that  $E_{u,v}$  has discrete mass points at the real poles, and for a real pole  $x_0$  we can rewrite (A.5) as

$$(A.6) \quad E_{u,v}(\{x_0\}) = \frac{1}{2\pi i} \oint_{(x_0)} \langle (x - L)^{-1}u, v \rangle dx$$

where the contour is taken in the subset where  $\langle (x - L)^{-1}u, v \rangle$  is meromorphic and such that it encircles only the pole  $x_0$ .

Finally, observe that from the explicit formula for the Green kernel (A.4) we get for  $x \in \mathbb{R}$  and  $\varepsilon > 0$ ,

$$(A.7) \quad \langle (x \pm i\varepsilon - L)^{-1}u, v \rangle = \sum_{k \leq l} \frac{\Phi_{x \pm i\varepsilon}(k) \overline{\phi_{x \mp i\varepsilon}(l)}}{[\Phi_{x \pm i\varepsilon}, \overline{\phi_{x \mp i\varepsilon}}]} (u(l) \overline{v(k)} + u(k) \overline{v(l)}) (1 - \frac{1}{2} \delta_{k,l}).$$

**A.2 The  $q$ -hypergeometric difference equation.** Consider the second-order  $q$ -difference equation, see [17, exerc. 1.13], in the following form

$$(A.8) \quad (y + y^{-1}) f(k) = (d - \frac{cq^{-k}}{dz}) f(k+1) + q^{-k} \frac{c+q}{dz} f(k) + (d^{-1} - \frac{q^{1-k}}{dz}) f(k-1).$$

We assume that  $d$  and  $z$  are non-zero, and as usual we take  $0 < q < 1$ . For the difference equation we have the following solutions in terms of basic hypergeometric series;

$$(A.9) \quad f_{\mu(y)}(k) = {}_2\varphi_1 \left( \begin{matrix} dy, d/y \\ c \end{matrix}; q, zq^k \right), \quad c \notin q^{-\mathbb{Z}_{\geq 0}}, \quad \mu(y) = \frac{1}{2}(y + y^{-1}),$$

which is symmetric in  $y$  and  $y^{-1}$  and

$$F_y(k) = (dy)^{-k} {}_2\varphi_1 \left( \begin{matrix} dy, qdy/c \\ qy^2 \end{matrix}; q, \frac{q^{1-k}c}{d^2z} \right), \quad y^2 \notin q^{-\mathbb{N}},$$

so that we also have  $F_{y^{-1}}(k)$  as a solution to (A.8). Here we use Jackson's transformation formula [17, (1.5.4)] to give a meaning to  $f_{\mu(y)}(k)$  and  $F_y(k)$  in case that  $|zq^k| \geq 1$  and  $|q^{1-k}c/d^2z| \geq 1$ , respectively, for  $z, d^2z/c \notin q^{\mathbb{Z}}$ .

These solutions are related by the expansion

$$(A.10) \quad \begin{aligned} f_{\mu(y)}(k) &= c(y)F_y(k) + c(y^{-1})F_{y^{-1}}(k), \\ c(x) &= \frac{(c/dy, d/y, dzy, q/dzy; q)_{\infty}}{(y^{-2}, c, z, q/z; q)_{\infty}}, \end{aligned}$$

for  $d, c, z \neq 0$ ,  $|\arg(-z)| < \pi$ ,  $c \notin q^{-\mathbb{Z}_{\geq 0}}$ ,  $y^2 \notin q^{\mathbb{Z}}$ , see [17, (4.3.2)] and use the theta-product identity (2.9).

Next we consider the associated operator

$$(A.11) \quad \xi_k \mapsto \left(d - \frac{cq^{-k}}{dz}\right) \xi_{k+1} + q^{-k} \frac{c+q}{dz} \xi_k + \left(d^{-1} - \frac{q^{1-k}}{dz}\right) \xi_{k-1},$$

where we now assume that  $\{\xi_k\}_{k \in \mathbb{Z}}$  is an orthogonal basis of  $\ell^2(\mathbb{Z})$ . We can now ask for what values of the parameters  $d$ ,  $c$  and  $z$  we can rewrite the operator as a symmetric operator of the form as in (A.1). Inserting the orthonormal basis  $e_k = \xi_k / \|\xi_k\|$  in (A.11) shows that we have to have that  $(c+q)/dz \in \mathbb{R}$  and

$$(A.12) \quad \frac{\|\xi_{k+1}\|^2}{\|\xi_k\|^2} = \frac{\bar{d}^{-1} - q^{-k}/\bar{d}\bar{z}}{d - cq^{-k}/dz} = \frac{1}{|d|^2} \frac{1 - q^{-k}/\bar{z}}{1 - cq^{-k}/d^2z}.$$

Hence the right hand side of (A.12) must be positive for all  $k \in \mathbb{Z}$ . Note that we assume that the numerator and the denominator are non-zero for all  $k \in \mathbb{Z}$ , in order not to reduce to the Jacobi matrix case. So we assume  $z, c/d^2z \notin q^{\mathbb{Z}}$ . On the other hand, if the right hand side of (A.12) is positive for all  $k \in \mathbb{Z}$  we can define  $\|\xi_k\|$  recursively from (A.12) and we find a symmetric operator of the form (A.1) assuming that  $(c+q)/dz \in \mathbb{R}$ . So positivity of the right hand side of (A.12) and  $(c+q)/dz \in \mathbb{R}$  are necessary and sufficient for the mapping in (A.11) to be symmetric. In the following lemma we give an explicit description of the parameter domain which satisfy these conditions. The proof is similar to the determination of unitary structures on irreducible principal series representations of  $U_q(\mathfrak{su}(1,1))$ , see [41, part II, §2].

**Lemma A.2.** *Assume  $z, c/d^2z \notin q^{\mathbb{Z}}$ . The right hand side of the  $q$ -hypergeometric difference equation (A.8) can be written as a symmetric operator on  $\ell^2(\mathbb{Z})$  of the form (A.1) if and only if  $(c+q)/dz \in \mathbb{R}$  and one of the following conditions holds: (1)  $\bar{z}c = d^2z$ , or (2)  $z > 0$ ,  $c \neq d^2$  and  $zq^{k_0+1} < c/d^2 < zq^{k_0}$ , where  $k_0 \in \mathbb{Z}$  is such that  $1 < q^{k_0}z < q^{-1}$ , or (3)  $z < 0$ ,  $c \neq d^2$  and  $c/d^2 > 0$ . In these cases the parameters of (A.1) are given by  $b_k = q^{-k}(c+q)/dz$  and*

$$a_k = \sqrt{\left(1 - \frac{q^{-k}}{z}\right)\left(1 - \frac{cq^{-k}}{d^2z}\right)},$$

after multiplying the basis  $\{e_k\}$  with suitable phase factors.

*Remark.* For later purposes, we furthermore assume that  $c, dz \in \mathbb{R}$ . For the cases (2) and (3) this implies  $c > 0$ ,  $d \in \mathbb{R}$ , while for case (1) this implies that  $c > 0$  and  $c = |d|^2$ , since

$|z|^2 = (dz)^2$ . Note that we may assume  $k_0 = 0$  by replacing  $k$  by  $k + k_0$  in (A.8), and replacing  $z$  by  $zq^{k_0}$ .

We now consider the cases described in Lemma A.2. The symmetric operators are given by  $2Le_k = a_ke_{k+1} + b_ke_k + a_{k-1}e_{k-1}$  with  $a_k$  and  $b_k$  as in Lemma A.2. Put

$$(A.13) \quad w(k) = e^{i\psi_k} |d|^k \sqrt{\frac{(cq^{1-k}/d^2z; q)_\infty}{(q^{1-k}/\bar{z}; q)_\infty}},$$

where  $\psi_k \in \mathbb{R}$  are such that  $\psi_{k+1} - \psi_k = \arg(d(1 - q^{-k}/\bar{z})) = \arg(d(1 - cq^{-k}/d^2z))$  for all  $k$ . Then  $u = wf = \sum_{k \in \mathbb{Z}} w(k)f(k)e_k$  is a solution to  $Lu = \mu(y)u$  if  $f(k)$  is a solution to the hypergeometric  $q$ -difference equation (A.8). Observe furthermore that for  $k \rightarrow -\infty$  we have  $|\arg(1 - q^{-k}/\bar{z})| = \mathcal{O}(q^{-k})$ , so that  $\psi_{k+1} - \psi_k \pmod{2\pi} \rightarrow \arg(d)$  as  $k \rightarrow -\infty$ .

**Lemma A.3.** *Let  $c, dz \in \mathbb{R}$  and assume that the parameters satisfy the conditions as described in Lemma A.2. Then the operator  $L$  with domain  $\mathcal{D}(\mathbb{Z})$  is essentially self-adjoint for  $0 < c \leq q^2$ .*

*Proof.* The  $a_k$  are bounded for  $k \rightarrow -\infty$ , so it suffices to show that the Jacobi matrix  $J^+$  associated to  $L$  is essentially self-adjoint, see the previous subsection. By [7, Ch. VII, §1, Thm. 1.4, Cor.] we have that  $J^+$  is essentially self-adjoint if  $a_k + a_{k-1} \pm b_k$  is bounded from above as  $k \rightarrow \infty$  for a choice of the sign. Use  $a_k = q^{-k} \sqrt{c/d^2z^2} - \frac{1}{2}(z + d^2z/c) + \mathcal{O}(q^k)$ ,  $k \rightarrow \infty$ , then the boundedness condition is satisfied if the coefficient of  $q^{-k}$  in  $a_k + a_{k-1} \pm b_k$  is non-positive. Since  $c > 0$ ,  $dz \in \mathbb{R}$ , this is the case when  $(1+q)\sqrt{c} \leq c+q$ . For  $0 < c \leq q^2$  the inequality holds.  $\square$

From now on, we will assume throughout this appendix that  $c, dz \in \mathbb{R}$ ,  $0 < c \leq q^2$ , and that the parameters satisfy the conditions as described in Lemma A.2. Let  $S_x^\pm$  be the eigenspaces of  $L$  corresponding to the eigenvalue  $x$  as defined in the previous subsection, and  $[\cdot, \cdot]$  the Wronskian associated to  $L$ .

**Lemma A.4.** *The solution  $wF_y$  spans  $S_{\mu(y)}^-$  for  $y \in \mathbb{C}$ ,  $|y| < 1$ , and  $wf_{\mu(y)}$  spans  $S_{\mu(y)}^+$  for  $\mu(y) \in \mathbb{C} \setminus \mathbb{R}$ . Furthermore,*

$$[wf_{\mu(y)}, wF_y] = \frac{1}{2}c(\bar{y}^{-1})(y - y^{-1})$$

when  $\mu(y) \in \mathbb{C} \setminus \mathbb{R}$ , where  $c(y)$  is defined in (A.10).

*Proof.* Since  $F_y(k) = \mathcal{O}((dy)^{-k})$  as  $k \rightarrow -\infty$ , the first statement follows from (A.13). Since  $f_{\mu(y)}(k) = \mathcal{O}(1)$  as  $k \rightarrow \infty$  and, by (2.9),

$$w(k) = e^{i\psi_k} |d|^k \left( \frac{c\bar{z}}{d^2z} \right)^{k/2} \sqrt{\frac{(\bar{z}q^k, d^2z/c, cq/d^2z; q)_\infty}{(d^2zq^k/c, \bar{z}, q/\bar{z}; q)_\infty}} \Rightarrow |w(k)| = \mathcal{O}(c^{\frac{1}{2}k}), \quad k \rightarrow \infty,$$

we have  $wf_{\mu(y)} \in S_{\mu(y)}^+$  for  $|c| < 1$ . By Lemma A.3 and the generalities of the previous subsection it follows that  $wf_{\mu(y)}$  spans the one-dimensional space  $S_{\mu(y)}^+$ .

It remains to calculate the Wronskian. By (A.10) and the fact that  $\overline{wF_y}$  is a constant multiple of  $wF_y$ , see §7.1, we have

$$[\overline{wf_{\mu(y)}}, wF_y] = \overline{c(\overline{y}^{-1})}[\overline{wF_{\overline{y}^{-1}}}, wF_y].$$

The lemma follows now from

$$\begin{aligned} [wF_y, \overline{wF_{\overline{y}^{-1}}}] &= \lim_{k \rightarrow -\infty} [wF_y, \overline{wF_{\overline{y}^{-1}}}] (k) = \\ &= \lim_{k \rightarrow -\infty} \frac{a_k}{2} |w(k)w(k+1)| (e^{i(\psi_{k+1}-\psi_k)} F_y(k+1) \overline{F_{\overline{y}^{-1}}(k)} - e^{i(\psi_k-\psi_{k+1})} F_y(k) \overline{F_{\overline{y}^{-1}}(k+1)}) \\ &= \lim_{k \rightarrow -\infty} \frac{1}{2} |d|^{2k+1} (e^{i \arg(d)} (yd)^{-k-1} (\overline{y}^{-1} d)^{-k} - e^{-i \arg(d)} (yd)^{-k} (\overline{y}^{-1} d)^{-k-1}) \\ &= \frac{1}{2} (y^{-1} - y). \quad \square \end{aligned}$$

We define for  $x \in \mathbb{C} \setminus \mathbb{R}$ ,  $\phi_x = wf_x$  and  $\Phi_x = wF_y$ , where  $y$  is the unique element in the open unit disk such that  $x = \mu(y)$ . By Lemma A.4 and Proposition A.1, we can give an expression of the resolvent  $(x - L)^{-1}$  in terms of the two functions  $\overline{\phi_x}$  and  $\Phi_x$ . In order to use (A.5) for the computation of the spectral measure of  $L$ , we have to calculate the limits as  $\varepsilon \downarrow 0$  in (A.7). Note that  $\phi_{x \pm i\varepsilon} \rightarrow wf_x$  as  $\varepsilon \downarrow 0$  for  $x \in \mathbb{R}$ . For the asymptotic solution  $\Phi_x$  we have to be more careful in computing the limit. For  $x \in \mathbb{R}$  satisfying  $|x| > 1$  we have  $\Phi_{x \pm i\varepsilon} \rightarrow wF_y$  as  $\varepsilon \downarrow 0$ , where  $y \in (-1, 1) \setminus \{0\}$  is such that  $\mu(y) = x$ . If  $t \in [-1, 1]$ , then we put  $x = \cos \chi = \mu(e^{i\chi})$  with  $\chi \in [0, \pi]$ , and then  $\Phi_{x-i\varepsilon} \rightarrow wF_{e^{i\chi}}$  and  $\Phi_{x+i\varepsilon} \rightarrow wF_{e^{-i\chi}}$  as  $\varepsilon \downarrow 0$ .

Let us for the moment assume that the zeros of the  $c$ -function of (A.10) are simple and do not coincide with its poles. For the case  $|x| < 1$ ,  $x = \cos \chi = \mu(e^{i\chi})$  and  $u, v \in \mathcal{D}(\mathbb{Z})$  we consider the limit

$$\begin{aligned} (A.14) \quad & \lim_{\varepsilon \downarrow 0} \langle (x - i\varepsilon - L)^{-1} u, v \rangle - \langle (x + i\varepsilon - L)^{-1} u, v \rangle \\ &= 2 \sum_{k \leq l} \left( \frac{w(k) F_{e^{i\chi}}(k) \overline{w(l) f_{\cos \chi}(l)}}{c(e^{i\chi})(e^{-i\chi} - e^{i\chi})} - \frac{w(k) F_{e^{-i\chi}}(k) \overline{w(l) f_{\cos \chi}(l)}}{c(e^{-i\chi})(e^{i\chi} - e^{-i\chi})} \right) \\ & \quad \times (u(l) \overline{v(k)} + u(k) \overline{v(l)}) (1 - \frac{1}{2} \delta_{k,l}). \end{aligned}$$

Observe that the term within the big brackets can be written in the following two ways,

$$\begin{aligned} (A.15) \quad & \frac{w(k) f_{\cos \chi}(k) \overline{w(l) f_{\cos \chi}(l)}}{|c(e^{i\chi})|^2 (e^{-i\chi} - e^{i\chi})} = \frac{w(k) F_{e^{i\chi}}(k) \overline{w(l) f_{\cos \chi}(l)}}{c(e^{i\chi})(e^{-i\chi} - e^{i\chi})} - \frac{w(k) F_{e^{-i\chi}}(k) \overline{w(l) f_{\cos \chi}(l)}}{c(e^{-i\chi})(e^{i\chi} - e^{-i\chi})} \\ &= \frac{1}{2} \left( \frac{w(k) f_{\cos \chi}(k) \overline{w(l) f_{\cos \chi}(l)}}{c(e^{i\chi}) c(e^{-i\chi}) [wF_{e^{-i\chi}}, wF_{e^{i\chi}}]} \right). \end{aligned}$$

Here the first equality follows from the connection coefficient formula (A.10) and the fact that  $|c(e^{i\chi})| = |c(e^{-i\chi})|$  by the conditions on the parameters. The second equality follows

again by the connection coefficient formula (A.10), and the fact that for  $y \in \mathbb{C} \setminus \{0\}$  with  $\mu(y) \in \mathbb{C} \setminus \mathbb{R}$ ,

$$\frac{wF_y}{[wF_y, \overline{wF_{\bar{y}^{-1}}}] = \frac{\overline{wF_{\bar{y}}}}{[\overline{wF_{\bar{y}}}, \overline{wF_{\bar{y}^{-1}}}] \Rightarrow \frac{2wF_y}{y^{-1} - y} = \frac{\overline{wF_{\bar{y}}}}{[wF_{\bar{y}}, wF_{\bar{y}^{-1}}]},$$

since  $\overline{wF_{\bar{y}}}$  is a constant multiple of  $wF_y$  and using the last step of the proof of Lemma A.4. From the second identity for the term in big brackets in (A.14) we see that it is symmetric in  $k$  and  $l$ , so we can symmetrise the sum in (A.14). Using then the first identity for the term in big brackets in (A.14) we obtain

$$\lim_{\varepsilon \downarrow 0} \langle (x - i\varepsilon - L)^{-1}u, v \rangle - \langle (x + i\varepsilon - L)^{-1}u, v \rangle = 2 \sum_{k,l=-\infty}^{\infty} \frac{\overline{w(k)f_{\cos \chi}(k)}u(k)w(l)f_{\cos \chi}(l)\overline{v(l)}}{|c(e^{i\chi})|^2(e^{-i\chi} - e^{i\chi})}.$$

Hence, with  $dx = \frac{i}{2}(e^{i\chi} - e^{-i\chi})d\chi$ , we obtain for  $0 \leq \chi_1 < \chi_2 \leq \pi$  and  $u, v \in \mathcal{D}(\mathbb{Z})$ ,

$$E_{u,v}((\cos \chi_2, \cos \chi_1)) = \frac{1}{2\pi} \int_{\chi_1}^{\chi_2} (\mathcal{F}u)(\cos \chi) \overline{(\mathcal{F}v)(\cos \chi)} \frac{d\chi}{|c(e^{i\chi})|^2},$$

where

$$(A.16) \quad (\mathcal{F}u)(x) = \langle u, wf_x \rangle = \sum_{k=-\infty}^{\infty} u(k) \overline{w(k)f_x(k)}$$

for  $u \in \mathcal{D}(\mathbb{Z})$  is the corresponding Fourier transform.

Next we consider the case  $|x| > 1$ ,  $x \in \mathbb{R}$ , then we have from (A.7) and Lemma A.4 that

$$\lim_{\varepsilon \downarrow 0} \langle (x \pm i\varepsilon - L)^{-1}u, v \rangle = 2 \sum_{k \leq l} \frac{w(k)F_y(k)\overline{w(l)f_{\mu(y)}(l)}}{c(y^{-1})(y^{-1} - y)} (u(l)\overline{v(k)} + u(k)\overline{v(l)}) (1 - \frac{1}{2}\delta_{k,l})$$

where  $u, v \in \mathcal{D}(\mathbb{Z})$  and where  $y \in (-1, 1) \setminus \{0\}$  is such that  $x = \mu(y)$ , provided that  $y^{-1}$  is not a zero of  $c(\cdot)$ . It follows by the bounded convergence theorem that  $E_{u,v}((x_1, x_2)) = 0$  when  $(x_1, x_2) \cap [-1, 1] = \emptyset$  and  $(x_1, x_2)$  does not contain  $x_0 = \mu(y_0)$  with  $y_0 \in (-1, 1)$  a zero of the map  $y \mapsto c(y^{-1})$ .

Suppose now that  $(x_1, x_2) \cap [-1, 1] = \emptyset$  and that  $(x_1, x_2)$  contains exactly one point  $x_0 = \mu(y_0)$  with  $c(y_0) = 0$ , where  $y_0 \in \mathbb{R}$  is such that  $|y_0| > 1$ . Suppose furthermore that  $y_0$  is a simple zero of  $c(\cdot)$ , and that  $c(y_0^{-1}) \neq 0$ . Then it follows from (A.6) after the change of variable  $x = \mu(y)$ , that

$$\langle E(\{x_0\})u, v \rangle = \langle E((x_1, x_2))u, v \rangle = \sum_{k \leq l} \text{Res}_{y=y_0^{-1}} \left( \frac{-1}{c(y^{-1})y} \right) w(k)F_{y_0^{-1}}(k)\overline{w(l)f_{x_0}(l)} (u(l)\overline{v(k)} + u(k)\overline{v(l)}) (1 - \frac{1}{2}\delta_{k,l}).$$

Now using the connection coefficient formula (A.10) and the fact that  $c(y_0) = 0$ , we have  $w(k)F_{y_0^{-1}}(k) = c(y_0^{-1})^{-1}w(k)f_{x_0}(k)$ . Since  $\overline{w(l)f_{x_0}(l)}w(k)f_{x_0}(k)$  is symmetric in  $k$  and  $l$ , cf. (A.15), we can symmetrise to find

$$\langle E(\{x_0\})u, v \rangle = \sum_{k,l=-\infty}^{\infty} \text{Res}_{y=y_0} \left( \frac{1}{c(y)c(y^{-1})y} \right) \overline{w(k)f_{x_0}(k)}u(k)w(l)f_{x_0}(l)\overline{v(l)}.$$

Observe that  $(z, q/z; q)_{\infty}c(y)$  is real for  $y \in \mathbb{R}$  and that all zeros of the  $c$ -function outside the unit disk are real. For parameters satisfying condition (2) or (3) of Lemma A.2 and  $c > 0$ ,  $dz \in \mathbb{R}$ , this is obvious. For parameters satisfying condition (1) of lemma A.2 and  $c > 0$  and  $dz \in \mathbb{R}$ , this follows from the fact that  $|d| = |c/d| < 1$  since  $|d|^2 = c \leq q^2$ . It follows now easily that, for generic parameters, the support of the resolution of the identity  $E$  of  $L$  is given by  $[-1, 1]$ , which is exactly the continuous spectrum of  $L$ , together with the discrete set  $\{x_0 = \mu(y_0) \mid y_0 \in \mathbb{R} \setminus [-1, 1], c(y_0) = 0\}$ , which is exactly the point spectrum of  $L$ , cf. [22] and [28]. These remarks prove a large part of the following theorem, see [22], [28] for more details.

**Theorem A.5.** *Consider  $d, z$  as non-zero complex parameters such that  $dz \in \mathbb{R}$ . Suppose that  $0 < c \leq q^2$ , and that  $z, c/d^2z \notin q^{\mathbb{Z}}$ . Assume furthermore that the parameters satisfy one of the following three conditions: (1)  $\bar{z}c = d^2z$ , or (2)  $z > 0$ ,  $c \neq d^2$  and  $zq^{k_0+1} < c/d^2 < zq^{k_0}$ , where  $k_0 \in \mathbb{Z}$  is such that  $1 < q^{k_0}z < q^{-1}$ , or (3)  $z < 0$ ,  $c \neq d^2$  and  $c/d^2 > 0$ . Consider the following unbounded operators on  $\ell^2(\mathbb{Z})$  defined initially on the domain  $\mathcal{D}$  of finite linear combinations of the orthonormal basis vectors  $\{e_k\}_{k \in \mathbb{Z}}$ :*

$$2L e_k = a_k e_{k+1} + b_k e_k + a_{k-1} e_{k-1},$$

$$b_k = q^{-k}(c+q)/dz \in \mathbb{R}, \quad a_k = \sqrt{(1 - \frac{q^{-k}}{z})(1 - \frac{cq^{-k}}{d^2z})} > 0.$$

*Then  $L$  is essentially self-adjoint, and the closure  $L^{**}$  of the operator  $L$  is given by the same formula on  $\mathcal{D}^{**}$ . The spectral decomposition  $L = \int_{\mathbb{R}} x dE(x)$  is given by*

$$\langle Lu, v \rangle = |(c, z, q/z; q)_{\infty}|^2 \int_{\mathbb{R}} x (\mathcal{F}u)(x) \overline{(\mathcal{F}v)(x)} d\nu(x; c/d, d; q/dz|q), \quad u \in \mathcal{D}^{**}, v \in \ell^2(\mathbb{Z}),$$

*where the measure  $d\nu(\cdot; a, b; d|q)$  is defined in (5.11), (5.5), and where the Fourier transform  $\mathcal{F} : \ell^2(\mathbb{Z}) \rightarrow L^2(\mathbb{R}, d\nu(\cdot; c/d, d; q/dz|q))$  is the unique continuous linear map which coincides with the formulas (A.16), (A.13) and (A.9) on  $\mathcal{D}$ .*

*Remark.* (i) This theorem extends the result by Takehi [21] to a much larger parameter set. Takehi's result corresponds to case (3) with  $z = -1$  and  $c$  and  $d$  in a discrete subset. The proof is essentially the same.

(ii) The Fourier transform  $\mathcal{F} : \ell^2(\mathbb{Z}) \rightarrow L^2(\mathbb{R}; d\nu(\cdot; c/d, d; dz|q))$  is in fact an isometric isomorphism after scaling it by  $(c, z, q/z; q)_{\infty}$ .

(iii) It can be shown that the closure of  $L$  has deficiency indices  $(1, 1)$  if one replaces the condition  $0 < c \leq q^2$  by  $q^2 < c < 1$ . Indeed, since  $|c| < 1$  we have  $wf_x \in S_x^+$  for  $x \in \mathbb{C} \setminus \mathbb{R}$ . On the other hand,

$$g_x(k) = q^k c^{-k} {}_2\varphi_1 \left( \begin{matrix} qdy/c, qd/cy \\ q^2/c \end{matrix}; q, zq^k \right), \quad x = \mu(y),$$

is also a solution of the  $q$ -hypergeometric difference equation (A.8). Since  $|w(k)g_x(k)| = \mathcal{O}(q^k |c|^{-k/2})$  as  $k \rightarrow \infty$ , we find a new  $\ell^2$ -solution of  $Lf = xf$  as  $k \rightarrow \infty$  for  $1 > c > q^2$ , which is linear independent of  $wf_x$ . So  $S_x^+$ ,  $x \in \mathbb{C} \setminus \mathbb{R}$ , is two-dimensional for  $q^2 < c < 1$ , which implies that  $L$  has deficiency indices  $(1, 1)$ .

## APPENDIX B. SUMMATION FORMULAS

### BY MIZAN RAHMAN

In this Appendix the proofs of Lemma 5.5 and Lemma 6.4 are given. In both cases it involves an expression for the Poisson kernel of the little  $q$ -Jacobi functions. The structure of the proof is similar in both cases. The proof of Lemma 5.5 splits into two cases; one for the absolutely continuous part and one for the infinite set of discrete mass points. This is treated in the first two subsections. The proof of Lemma 6.4, treated in the last subsection, is similar to, but simpler than, the proof of Lemma 5.5 for the absolutely continuous case.

**B.1 Proof of Lemma 5.5 for the absolutely continuous part.** The idea of the proof is to write the product of two little  $q$ -Jacobi functions as an infinite sum of Askey-Wilson polynomials, and next to use an integral representation for the Askey-Wilson polynomials. Interchanging summation and integration gives a summable series as the integrand. The resulting integral can then be evaluated, and after some series manipulation we arrive at the desired result. We give the proof in several steps. Recall that our basic assumption is that the real parameters  $s$  and  $t$  satisfy  $|t| > 1$ ,  $|s| > 1$ .

First use [17, (1.4.6)] to write the little  $q$ -Jacobi function of §5.2 as

$$\phi_n(x; s, t|q^2) = \frac{(-q^{2n}t^{-2}; q^2)_\infty}{{(-q^{2n}; q^2)_\infty}} {}_2\varphi_1 \left( \begin{matrix} qtz/s, qt/sz \\ q^2s^{-2} \end{matrix}; q^2, -q^{2n}t^{-2} \right), \quad x = \mu(z) = \frac{1}{2}(z + z^{-1}),$$

where we use the analytic continuation of the  ${}_2\varphi_1$ -series as in [17, Ch. 4]. Using the theta-product identity (2.9) we see that we have to evaluate

$$\begin{aligned} (B.1) \quad & \sum_{n=-\infty}^{\infty} \left( \frac{uq^2}{s^2} \right)^n {}_2\varphi_1 \left( \begin{matrix} qte^{i\theta}/s, qte^{-i\theta}/s \\ q^2s^{-2} \end{matrix}; q^2, -q^{2n}t^{-2} \right) {}_2\varphi_1 \left( \begin{matrix} qe^{i\theta}/ts, qe^{-i\theta}/ts \\ q^2s^{-2} \end{matrix}; q^2, -q^{2n} \right) \\ & = Q_u(\cos \theta) = \frac{(-1, -q^2; q^2)_\infty}{{(-q^2t^2, -t^{-2}; q^2)_\infty}} R_u(\cos \theta; s, t|q^2). \end{aligned}$$

Recall the definition of the Askey-Wilson polynomials, see [5], [17, §7.5],

$$(B.2) \quad p_m(x; a, b, c, d|q) = {}_4\varphi_3 \left( \begin{matrix} q^{-m}, q^{m-1}abcd, ax, a/x \\ ab, ac, ad \end{matrix}; q, q \right).$$

We can take the first step, which allows us to separate the summation variable  $n$  from the product of the two  ${}_2\varphi_1$ -series in (B.1).



**Lemma B.1.** *For  $|w| < 1$  we have*

$${}_2\varphi_1 \left( \begin{matrix} qte^{i\theta}/s, qte^{-i\theta}/s \\ q^2s^{-2} \end{matrix}; q^2, wt^{-2} \right) {}_2\varphi_1 \left( \begin{matrix} qe^{i\theta}/st, qe^{-i\theta}/st \\ q^2s^{-2} \end{matrix}; q^2, w \right) = \sum_{m=0}^{\infty} \frac{(q^2s^{-2}e^{2i\theta}; q^2)_m}{(q^2; q^2)_m} (wq^{-1}se^{-i\theta}t^{-1})^m p_m(t; \frac{qe^{i\theta}}{s}, \frac{qe^{i\theta}}{s}, \frac{qe^{-i\theta}}{s}, \frac{qe^{-i\theta}}{s} | q^2).$$

*Proof.* Since  $|w| < 1$  and  $|t| > 1$ , we have  $|wt^{-2}| < 1$ . Use the series representation of the two  ${}_2\varphi_1$ -series to write the left hand side as an absolutely convergent double sum. Next split off the power of  $w$  in order to write the left hand side as

$$\sum_{m=0}^{\infty} w^m \sum_{k=0}^m \frac{(qe^{i\theta}/st, qe^{-i\theta}/st; q^2)_{m-k}}{(q^2s^{-2}, q^2; q^2)_{m-k}} \frac{(qte^{i\theta}/s, qte^{-i\theta}/s; q^2)_k}{(q^2s^{-2}, q^2; q^2)_k} t^{-2k}.$$

Using elementary relations for the  $q$ -shifted factorials, see [17, §1.2], we can rewrite this as

$$\sum_{m=0}^{\infty} w^m \frac{(qe^{i\theta}/st, qe^{-i\theta}/st; q^2)_m}{(q^2s^{-2}, q^2; q^2)_m} {}_4\varphi_3 \left( \begin{matrix} q^{-2m}, s^2q^{-2m}, qte^{i\theta}/s, qte^{-i\theta}/s \\ q^2s^{-2}, q^{1-2m}ste^{i\theta}, q^{1-2m}ste^{-i\theta} \end{matrix}; q^2, q^2 \right).$$

Since the  ${}_4\varphi_3$ -series is terminating and balanced we can transform it using Sears's transformation [17, (2.10.4)] with  $a, d$  replaced by  $qte^{i\theta}/s, q^2s^{-2}$ . Then the  ${}_4\varphi_3$ -series can be written as an Askey-Wilson polynomial, and keeping track of the constants proves the lemma.  $\square$

Our next step is to use an integral representation for the Askey-Wilson polynomial in Lemma B.1. There is a number of ( $q$ -)integrals for the Askey-Wilson polynomial available.

**Lemma B.2.** *We have the integral representation for the Askey-Wilson polynomial;*

$$p_m(t; \frac{qe^{i\theta}}{s}, \frac{qe^{i\theta}}{s}, \frac{qe^{-i\theta}}{s}, \frac{qe^{-i\theta}}{s} | q^2) = A \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(\frac{q\sigma}{s}e^{-i(\theta+\psi)}; q^2)_m}{(\frac{q^3}{\sigma s^3}e^{i(\theta+\psi)}; q^2)_m} \left( \frac{qe^{i(\theta+\psi)}}{\sigma s} \right)^m \\ \times \frac{(\frac{kt}{\sigma}e^{i\psi}, \frac{q^2\sigma}{kt}e^{-i\psi}, k\sigma te^{-i\psi}, \frac{q^2}{k\sigma t}e^{i\psi}, \frac{q^3}{\sigma s^3}e^{i(\theta+\psi)}; q^2)_{\infty}}{(\frac{q}{\sigma s}e^{i(\theta+\psi)}, \frac{q}{\sigma s}e^{i(\theta+\psi)}, \frac{q}{\sigma s}e^{i(\psi-\theta)}, \sigma te^{-i\psi}, \frac{\sigma}{t}e^{-i\psi}; q^2)_{\infty}} d\psi, \\ A = \frac{(q^2, qte^{i\theta}/s, qe^{i\theta}/st, qte^{-i\theta}/s, qe^{-i\theta}/st, qte^{i\theta}/s, qe^{i\theta}/st; q^2)_{\infty}}{(k, q^2/k, t^2k, q^2/t^2k, q^2e^{2i\theta}/s^2, q^2s^{-2}, q^2s^{-2}; q^2)_{\infty}},$$

where  $\sigma$  and  $k$  are free parameters such that there are no zeros in the denominator.

*Proof.* We start with the integral representation for a very-well-poised  ${}_8\varphi_7$ -series given in [17, Exerc. 4.4, p. 122], which can be proved by a residue calculation. We use [17, Exerc. 4.4, p. 122] in base  $q^2$  and with the parameters  $a = g = qe^{i\theta}/\sigma s$ ,  $b = qe^{-i\theta}/\sigma s$ ,

$c = \sigma t$ ,  $d = \sigma/t$ ,  $f = sq^{1-2m}e^{i\theta}/\sigma$  and  $h = sqe^{i\theta}/\sigma$  where  $\sigma$  and  $k$  are free parameters. This gives an integral representation for a terminating very-well-poised  ${}_8\varphi_7$ -series;

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(\frac{kt}{\sigma}e^{i\psi}, \frac{q^2\sigma}{kt}e^{-i\psi}, k\sigma te^{-i\psi}, \frac{q^2}{k\sigma t}e^{i\psi}, \frac{q^{3+2m}}{\sigma s^3}e^{i(\theta+\psi)}, \frac{s}{\sigma}q^{1-2m}e^{i(\theta+\psi)}; q^2)_{\infty}}{(\frac{q}{\sigma s}e^{i(\theta+\psi)}, \frac{q}{\sigma s}e^{i(\theta+\psi)}, \frac{q}{\sigma s}e^{i(\psi-\theta)}, \sigma te^{-i\psi}, \frac{\sigma}{t}e^{-i\psi}, \frac{sq}{\sigma}e^{i(\theta+\psi)}; q^2)_{\infty}} d\psi = \\ \frac{(k, \frac{q^2}{k}, t^2k, \frac{q^2}{t^2k}, \frac{q^2}{s^2}e^{2i\theta}, \frac{q^2}{s^2}, ste^{i\theta}q^{1-2m}, \frac{s}{t}e^{i\theta}q^{1-2m}, q^2e^{2i\theta}, \frac{q^{2+2m}}{s^2}; q^2)_{\infty}}{(q^2, q\frac{t}{s}e^{i\theta}, \frac{q}{st}e^{i\theta}, q\frac{t}{s}e^{-i\theta}, \frac{q}{st}e^{-i\theta}, q\frac{t}{s}e^{i\theta}, \frac{q}{st}e^{i\theta}, qste^{i\theta}, q\frac{s}{t}e^{i\theta}, q^{2-2m}e^{2i\theta}; q^2)_{\infty}} \\ \times {}_8W_7(q^{-2m}e^{2i\theta}; s^2q^{-2m}, s^2q^{-2m}e^{2i\theta}, qte^{i\theta}/s, qe^{i\theta}/st, q^{-2m}; q^2, q^{2+2m}s^{-2}). \end{aligned}$$

Now we can apply Watson's formula [17, (2.5.1)] with  $d = qte^{i\theta}/s$ ,  $e = qe^{i\theta}/st$  to transform the terminating very-well-poised  ${}_8\varphi_7$ -series into a terminating balanced  ${}_4\varphi_3$ -series. This shows that the  ${}_8W_7$ -series equals

$$\frac{(q^{2-2m}e^{2i\theta}, s^2q^{-2m}; q^2)_m}{(q^{1-2m}se^{i\theta}/t, q^{1-2m}ste^{i\theta}; q^2)_m} p_m(t; \frac{qe^{i\theta}}{s}, \frac{qe^{i\theta}}{s}, \frac{qe^{-i\theta}}{s}, \frac{qe^{-i\theta}}{s} | q^2),$$

the desired Askey-Wilson polynomial. Collecting the results proves the lemma.  $\square$

In the proof the freedom to choose  $k$  wisely is crucial, but the  $\sigma$ -dependence is not essential. Combining Lemmas B.1 and B.2 gives the following expression

$$\begin{aligned} (B.3) \quad {}_2\varphi_1 \left( \frac{qte^{i\theta}/s, qte^{-i\theta}/s}{q^2s^{-2}}; q^2, wt^{-2} \right) {}_2\varphi_1 \left( \frac{qe^{i\theta}/st, qe^{-i\theta}/st}{q^2s^{-2}}; q^2, w \right) = \\ A \frac{1}{2\pi} \int_{-\pi}^{\pi} B(e^{i\psi}) {}_2\varphi_1 \left( \frac{q^2s^{-2}e^{2i\theta}, \frac{q\sigma}{s}e^{-i(\theta+\psi)}}{\frac{q^3}{\sigma s^3}e^{i(\theta+\psi)}}; q^2, \frac{we^{i\psi}}{\sigma t} \right) d\psi, \\ B(e^{i\psi}) = \frac{(\frac{kt}{\sigma}e^{i\psi}, \frac{q^2\sigma}{kt}e^{-i\psi}, k\sigma te^{-i\psi}, \frac{q^2}{k\sigma t}e^{i\psi}, \frac{q^3}{\sigma s^3}e^{i(\theta+\psi)}; q^2)_{\infty}}{(\frac{q}{\sigma s}e^{i(\theta+\psi)}, \frac{q}{\sigma s}e^{i(\theta+\psi)}, \frac{q}{\sigma s}e^{i(\psi-\theta)}, \sigma te^{-i\psi}, \frac{\sigma}{t}e^{-i\psi}; q^2)_{\infty}} \end{aligned}$$

and  $A$  as in Lemma B.2. For  $|w/\sigma t| < 1$  interchanging integration and summation is justified. Note that (B.3) also gives the analytic extension of the left hand side to  $w \in \mathbb{C} \setminus [1, \infty)$  using the analytic continuation of the  ${}_2\varphi_1$ -series in the integrand, e.g. using [17, (1.4.4)],

$$\begin{aligned} (B.4) \quad {}_2\varphi_1 \left( \frac{q^2s^{-2}e^{2i\theta}, \frac{q\sigma}{s}e^{-i(\theta+\psi)}}{\frac{q^3}{\sigma s^3}e^{i(\theta+\psi)}}; q^2, \frac{we^{i\psi}}{\sigma t} \right) = \\ \frac{(q^2e^{2i\theta}/s^2, wqe^{-i\theta}/st; q^2)_{\infty}}{(q^3e^{i(\theta+\psi)}/\sigma s^3, we^{i\psi}/\sigma t; q^2)_{\infty}} {}_2\varphi_1 \left( \frac{qe^{i(\psi-\theta)}/\sigma s, we^{i\psi}/\sigma t}{wqe^{-i\theta}/st}; q^2, q^2e^{2i\theta}s^{-2} \right). \end{aligned}$$

We use (B.3) with (B.4) in (B.1) and we interchange summation and integration, which is easily justified. Then we have to evaluate a sum where now the summand consists of one  ${}_2\varphi_1$ -series. This is done in the following lemma.

**Lemma B.3.** For  $\max(1, |s\sigma/q|) < |u| < s^2q^{-2}$  we have

$$\sum_{n=-\infty}^{\infty} (uq^2s^{-2})^n {}_2\varphi_1 \left( \begin{matrix} q^2s^{-2}e^{2i\theta}, \frac{q\sigma}{s}e^{-i(\theta+\psi)} \\ \frac{q^3}{\sigma s^3}e^{i(\theta+\psi)} \end{matrix}; q^2, \frac{-q^{2n}e^{i\psi}}{\sigma t} \right) = \frac{(q^2, q^2e^{2i\theta}/s^2, qe^{i(\theta+\psi)}/\sigma su, q\sigma e^{-i(\theta+\psi)}/s, -\sigma ts^2e^{-i\psi}/u, -uq^2e^{i\psi}/\sigma ts^2; q^2)_{\infty}}{(e^{2i\theta}/u, uq^2/s^2, \sigma se^{-i(\psi+\theta)}/qu, q^3e^{i(\theta+\psi)}/\sigma s^3, -e^{i\psi}/\sigma t, -q^2\sigma te^{-i\psi}; q^2)_{\infty}}$$

*Proof.* Use the analytic continuation of (B.4) and interchange summation to write the left hand side as

$$\frac{(q^2e^{2i\theta}s^{-2}; q^2)_{\infty}}{(q^3e^{i(\theta+\psi)}/\sigma s^3; q^2)_{\infty}} \sum_{m=0}^{\infty} \frac{(qe^{i(\psi-\theta)}/\sigma s; q^2)_m}{(q^2; q^2)_m} \left( \frac{q^2e^{2i\theta}}{s^2} \right)^m \times \sum_{n=-\infty}^{\infty} \frac{(-q^{1+2n+2m}e^{-i\theta}/st; q^2)_{\infty}}{(-q^{2n+2m}e^{i\psi}/\sigma t; q^2)_{\infty}} \left( \frac{uq^2}{s^2} \right)^n.$$

The inner sum can be evaluated by Ramanujan's  ${}_1\psi_1$ -summation formula [17, (5.2.1)] for  $|s\sigma/q| < |u| < |s^2q^{-2}|$ . The dependence on  $m$  of the result is easy using the theta-product identity (2.9). Explicitly, the inner sum equals

$$\left( \frac{s^2}{uq^2} \right)^m \frac{(q^2, q\sigma e^{-i(\theta+\psi)}/s, -uq^2e^{i\psi}/\sigma ts^2, -e^{-i\psi}\sigma ts^2/u; q^2)_{\infty}}{(uq^2/s^2, \sigma se^{-i(\psi+\theta)}/qu, -e^{i\psi}/\sigma t, -q^2\sigma te^{-i\psi}; q^2)_{\infty}}.$$

Then the inner sum to be evaluated reduces to

$$\sum_{m=0}^{\infty} \frac{(qe^{i(\psi-\theta)}/\sigma s; q^2)_m}{(q^2; q^2)_m} \left( \frac{e^{2i\theta}}{u} \right)^m = \frac{(qe^{i(\psi+\theta)}/\sigma su; q^2)_{\infty}}{(e^{2i\theta}/u; q^2)_{\infty}}$$

by [17, (1.3.2)] for  $|u| > 1$ . Collecting the intermediate results gives the lemma.  $\square$

Combining (B.3) and Lemma B.3 gives an integral representation for  $Q_u(\cos \theta)$  on  $[-\pi, \pi]$  where the integrand consists of a quotient of eight infinite  $q$ -shifted factorials in the numerator and denominator. If we specialise one of the free parameters,  $k = -q^2$ , this reduces to six infinite products in the nominator and denominator. Explicitly, (B.5)

$$Q_u(\cos \theta) = \frac{B}{2\pi} \int_{-\pi}^{\pi} \frac{(-q^2te^{i\psi}, \frac{-\sigma e^{-i\psi}}{t}, \frac{qe^{i(\psi+\theta)}}{\sigma su}, \frac{q\sigma e^{-i(\theta+\psi)}}{s}, \frac{-\sigma ts^2e^{-i\psi}}{u}, \frac{-uq^2e^{i\psi}}{\sigma ts^2}; q^2)_{\infty}}{(\frac{qe^{i(\psi+\theta)}}{\sigma s}, \frac{qe^{i(\psi+\theta)}}{\sigma s}, \frac{qe^{i(\psi-\theta)}}{\sigma s}, \sigma te^{-i\psi}, \frac{\sigma}{t}e^{-i\psi}, \frac{\sigma se^{-i(\theta+\psi)}}{qu}; q^2)_{\infty}} d\psi, \\ B = \frac{(q^2, q^2, qte^{i\theta}/s, qe^{i\theta}/st, qte^{-i\theta}/s, qe^{-i\theta}/st, qte^{i\theta}/s, qe^{i\theta}/st; q^2)_{\infty}}{(-1, -q^2, -q^2t^2, -t^{-2}, e^{2i\theta}/u, q^2s^{-2}, q^2s^{-2}, uq^2s^{-2}; q^2)_{\infty}}$$

The integral in (B.5) is of the type considered in [17, §§4.9-10] in base  $q^2$  meaning that we can evaluate the integral by residue calculus. Indeed we may apply [17, (4.10.8)] with the specialisation  $A = B = C = D = 3$ ,  $m = 0$  and the twelve other parameters given by

$a_1 = -q^2 t / \sigma$ ,  $a_2 = q e^{i\theta} / \sigma s u$ ,  $a_3 = -u q^2 / \sigma t s^2$ ,  $b_1 = -\sigma / t$ ,  $b_2 = q \sigma e^{-i\theta} / s$ ,  $b_3 = -\sigma t s^2 / u$ ,  $c_2 = c_1 = q e^{i\theta} / \sigma s$ ,  $c_3 = q e^{-i\theta} / \sigma s$ ,  $d_1 = \sigma t$ ,  $d_2 = \sigma / t$ ,  $d_3 = \sigma s e^{-i\theta} / q u$ . Then the condition [17, (4.10.2)] is trivially satisfied, so that we may apply [17, (4.10.8)] to write the integral (B.5) as a sum of three  ${}_6\varphi_5$ -series. Due to  $a_1 b_1 = q^2$ ,  $a_3 b_3 = q^2$ , these  ${}_6\varphi_5$ -series reduce to  ${}_4\varphi_3$ -series. So we have written the integral in (B.5) as a sum of three  ${}_4\varphi_3$ -series in this way. If we further specialise  $u = q^{-2}$  two of these  ${}_4\varphi_3$ -series reduce to  ${}_3\varphi_2$ -series. The result is independent of  $\sigma$  and it reads

(B.6)

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\left( \frac{-q^2 t e^{i\psi}}{\sigma}, \frac{-\sigma e^{-i\psi}}{t}, \frac{q^3 e^{i(\psi+\theta)}}{\sigma s}, \frac{q \sigma e^{-i(\theta+\psi)}}{s}, -\sigma t s^2 q^2 e^{-i\psi}, \frac{-e^{i\psi}}{\sigma t s^2}; q^2 \right)_{\infty}}{\left( \frac{q e^{i(\psi+\theta)}}{\sigma s}, \frac{q e^{i(\psi+\theta)}}{\sigma s}, \frac{q e^{i(\psi-\theta)}}{\sigma s}, \sigma t e^{-i\psi}, \frac{\sigma e^{-i\psi}}{t}, \sigma s q e^{-i(\theta+\psi)}; q^2 \right)_{\infty}} d\psi = \\ & \frac{(-q^2 t^2, \frac{q^3 t e^{i\theta}}{s}, -s^{-2}, -t^{-2}, \frac{q e^{-i\theta}}{s t}, -q^2 s^2; q^2)_{\infty}}{(q^2, \frac{q t e^{i\theta}}{s}, \frac{q t e^{i\theta}}{s}, \frac{q t e^{-i\theta}}{s}, t^{-2}, \frac{q s e^{-i\theta}}{t}; q^2)_{\infty}} {}_3\varphi_2 \left( \begin{matrix} \frac{q t e^{i\theta}}{s}, \frac{q t e^{-i\theta}}{s}, q s t e^{i\theta} \\ q^2 t^2, \frac{q^3 t e^{i\theta}}{s} \end{matrix}; q^2, q^2 \right) \\ & + \frac{(-q^2, \frac{q^3 e^{i\theta}}{s t}, -t^{-2} s^{-2}, -1, q \frac{t}{s} e^{-i\theta}, -q^2 s^2 t^2; q^2)_{\infty}}{(q^2, \frac{q}{s t} e^{i\theta}, \frac{q}{s t} e^{i\theta}, \frac{q}{s t} e^{-i\theta}, t^2, q t s e^{-i\theta}; q^2)_{\infty}} {}_3\varphi_2 \left( \begin{matrix} \frac{q}{s t} e^{i\theta}, \frac{q}{s t} e^{-i\theta}, q \frac{s}{t} e^{i\theta} \\ \frac{q^3}{s t} e^{i\theta}, \frac{q^2}{t^2} \end{matrix}; q^2, q^2 \right) \\ & + \frac{(-q^3 s t e^{-i\theta}, q^4, \frac{-q e^{-i\theta}}{s t}, \frac{-e^{i\theta}}{s t q}, s^{-2}, -s t q e^{i\theta}; q^2)_{\infty}}{(q^2, q^2, q^2, q^2 e^{-2i\theta}, \frac{t e^{i\theta}}{q s}, \frac{e^{i\theta}}{s t q}; q^2)_{\infty}} {}_4\varphi_3 \left( \begin{matrix} q^2, q^2, q^2 e^{-2i\theta}, q^2 s^2 \\ q^4, q^3 \frac{s}{t} e^{-i\theta}, q^3 s t e^{-i\theta} \end{matrix}; q^2, q^2 \right). \end{aligned}$$

Combining (B.6) with (B.5) and (B.1) gives an expression for  $R_{q^{-2}}(\cos \theta; s, t | q^2)$  in terms of two  ${}_3\varphi_2$ -series and one  ${}_4\varphi_3$ -series. In order to bring the very-well-poised  ${}_8\varphi_7$ -series into play we use Bailey's extension of Watson's transformation formula [17, (2.10.10)] in base  $q^2$  with parameters  $a = q^2 t^2$ ,  $b = q s t e^{-i\theta}$ ,  $c = q^2$ ,  $d = q t e^{i\theta} / s$ ,  $e = q t e^{-i\theta} / s$ ,  $f = q s t e^{i\theta}$ . This gives the possibility to write the  ${}_4\varphi_3$ -series in (B.6) as the sum of a very-well-poised series as in Lemma 5.5 and a  ${}_3\varphi_2$ -series with the same parameters as the first  ${}_3\varphi_2$ -series on the right hand side of (B.6);

(B.7)

$$\begin{aligned} & \frac{(-q^3 s t e^{-i\theta}, q^4, \frac{-q e^{-i\theta}}{s t}, \frac{-e^{i\theta}}{s t q}, s^{-2}, -s t q e^{i\theta}; q^2)_{\infty}}{(q^2, q^2, q^2, q^2 e^{-2i\theta}, \frac{t}{s q} e^{i\theta}, \frac{e^{i\theta}}{s t q}; q^2)_{\infty}} {}_4\varphi_3 \left( \begin{matrix} q^2, q^2, q^2 e^{-2i\theta}, q^2 s^2 \\ q^4, q^3 \frac{s}{t} e^{-i\theta}, q^3 s t e^{-i\theta} \end{matrix}; q^2, q^2 \right) = \\ & \frac{(q^2 t^2, s^{-2}, q^3 s t e^{i\theta}, q^3 t e^{i\theta} / s, q^3 t e^{-i\theta} / s, -s t q^3 e^{-i\theta}, -e^{i\theta} / q s t, -q s t e^{i\theta}, -q e^{-i\theta} / s t; q^2)_{\infty}}{(q^2, q^2, q^4 t^2, q^2 e^{-2i\theta}, q t e^{i\theta} / s, q t e^{-i\theta} / s, q s t e^{i\theta}, q t e^{i\theta} / s, e^{i\theta} / q s t; q^2)_{\infty}} \\ & \quad \times {}_8W_7(q^2 t^2; q t e^{i\theta} / s, q t e^{-i\theta} / s, q s t e^{i\theta}, q s t e^{-i\theta}, q^2; q^2, q^2) \\ & - \frac{(q^2 t^2, q^3 t e^{i\theta} / s, q^2 s^2, s^{-2}, -e^{i\theta} / q s t, -s t q^3 e^{-i\theta}, -q s t e^{i\theta}, -q e^{-i\theta} / s t; q^2)_{\infty}}{(q^2, q^3 s t e^{-i\theta}, q s e^{-i\theta} / t, q t e^{i\theta} / s, q t e^{i\theta} / s, q t e^{-i\theta} / s, q s t e^{i\theta}, e^{i\theta} / q s t; q^2)_{\infty}} \\ & \quad \times {}_3\varphi_2 \left( \begin{matrix} q t e^{i\theta} / s, q t e^{-i\theta} / s, q s t e^{i\theta} \\ q^2 t^2, q^3 t e^{i\theta} / s \end{matrix}; q^2, q^2 \right). \end{aligned}$$

If we now use (B.7) in (B.6) and (B.5) we obtain the  ${}_8W_7$ -series on the right hand side of Lemma 5.5 with the factor in front using straightforward manipulations of  $q$ -shifted factorials.

It remains to show that the remaining terms can be summed explicitly. For this we first consider the factor in front of the first  ${}_3\varphi_2$ -series in (B.6) after having plugged in (B.7) for the  ${}_4\varphi_3$ -series. This factor is

$$\begin{aligned} & \frac{(q^3 t e^{i\theta}/s; q^2)_\infty}{(q^2, q t e^{i\theta}/s, q t e^{i\theta}/s, q t e^{-i\theta}/s, q s e^{-i\theta}/t, q s t e^{i\theta}, q^3 s t e^{-i\theta}, e^{i\theta}/q s t, t^{-2}; q^2)_\infty} \\ & \left( (-q^2 t^2, -t^{-2}, -s^{-2}, -q^2 s^2, \frac{q e^{-i\theta}}{s t}, q s t e^{i\theta}, q^3 s t e^{-i\theta}, \frac{e^{i\theta}}{q s t}; q^2)_\infty - \right. \\ & \left. (q^2 t^2, t^{-2}, s^{-2}, q^2 s^2, \frac{-q e^{-i\theta}}{s t}, -q s t e^{i\theta}, -q^3 s t e^{-i\theta}, \frac{-e^{i\theta}}{q s t}; q^2)_\infty \right) \\ & = \frac{(q^3 t e^{i\theta}/s, q s e^{i\theta}/t, -1, -q^2, -q^2 s^2 t^2, -s^{-2} t^{-2}; q^2)_\infty}{(q t e^{i\theta}/s, q s t e^{i\theta}, q s t e^{-i\theta}, q e^{i\theta}/s t, q^2, t^{-2}; q^2)_\infty} \end{aligned}$$

where we used the theta-product identity (5.16) with  $\lambda = q s t$ ,  $\mu = -e^{i\theta}$ ,  $x = -q t/s$  and  $\nu = -q s t$ , see [17, Exerc. 2.16]. Having used this identity we see that the resulting sum of two  ${}_3\varphi_2$ -series in (B.6) after having applied (B.7) can be summed by the non-terminating version of the Saalschütz summation formula [17, (2.10.12)] with  $e = q^2 t^2$ ,  $f = t q^3 e^{i\theta}/s$  in the form

$$\begin{aligned} & \frac{(t^{-2}, q \frac{t}{s} e^{i\theta}, q \frac{t}{s} e^{-i\theta}, q s t e^{i\theta}, \frac{q^3}{s t} e^{i\theta}; q^2)_\infty}{(t^2, \frac{q}{s t} e^{i\theta}, \frac{q}{s t} e^{-i\theta}, q \frac{s}{t} e^{i\theta}, q^3 \frac{t}{s} e^{i\theta}; q^2)_\infty} {}_3\varphi_2 \left( \begin{matrix} \frac{q}{s t} e^{i\theta}, \frac{q}{s t} e^{-i\theta}, q \frac{s}{t} e^{i\theta} \\ \frac{q^3}{s t} e^{i\theta}, \frac{q^2}{t^2} \end{matrix}; q^2, q^2 \right) + \\ & {}_3\varphi_2 \left( \begin{matrix} \frac{q t e^{i\theta}}{s}, \frac{q t e^{-i\theta}}{s}, q s t e^{i\theta} \\ q^2 t^2, \frac{q^3 t e^{i\theta}}{s} \end{matrix}; q^2, q^2 \right) = \frac{(t^{-2}, q^2, q^2 e^{2i\theta}, q^2/s^2; q^2)_\infty}{(\frac{q}{s t} e^{i\theta}, \frac{q}{s t} e^{-i\theta}, q \frac{s}{t} e^{i\theta}, q^3 \frac{t}{s} e^{i\theta}; q^2)_\infty}. \end{aligned}$$

This gives the term with  $q$ -shifted factorials in Lemma 5.5.  $\square$

**B.2 Proof of Lemma 5.5 for the infinite set of discrete mass points.** Since the radius of convergence for  $u$  of  $R_u(x; s, t|q^2)$  depends on  $x$ , we cannot use the result obtained for  $x \in [-1, 1]$  in the previous subsection to obtain the value for  $x = \mu(q^{1-2k} s t)$ , cf. [31, §6]. In order to prove Lemma 5.5 for the infinite set of discrete mass points we take  $x = \mu(q^{1-2k} s t)$  in the definition of  $R_u(x; s, t|q^2)$  in Proposition 5.4. For this argument we can use [17, (1.4.4), (1.4.5)] to rewrite the little  $q$ -Jacobi function as

$$\begin{aligned} & {}_2\varphi_1 \left( \begin{matrix} -q^{2-2k}, -q^{2k} s^{-2} t^{-2} \\ q^2 s^{-2} \end{matrix}; q^2, -q^{2n} \right) \\ & = \frac{(-q^{2k} s^{-2} t^{-2}, q^{2+2n-2k}; q^2)_\infty}{(q^2 s^{-2}, -q^{2n}; q^2)_\infty} {}_2\varphi_1 \left( \begin{matrix} -q^{2-2k} t^2, -q^{2n} \\ q^{2+2n-2k} \end{matrix}; q^2, -\frac{q^{2k}}{s^2 t^2} \right) \\ & = \frac{(-q^{2k} s^{-2}, q^{2+2n-2k}; q^2)_\infty}{(q^2 s^{-2}, -q^{2n}; q^2)_\infty} {}_2\varphi_1 \left( \begin{matrix} -q^{2n} t^{-2}, -q^{2-2k} \\ q^{2+2n-2k} \end{matrix}; q^2, -\frac{q^{2k}}{s^2} \right), \end{aligned}$$

This shows the  $q$ -Bessel coefficient behaviour of the little  $q$ -Jacobi function at these discrete mass points. In case the absolute value of the argument is greater than one we can use

Jackson's transformation of a  ${}_2\varphi_1$ -series to a  ${}_2\varphi_2$ -series, see [17, (1.5.4)], to give the analytic extension which respects the  $q$ -Bessel coefficient behaviour. Hence, we find

$$(B.8) \quad R_u(\mu(-stq^{1-2k}); s, t|q^2) = \frac{(-t^{-2}, -q^2t^2, -q^{2k}s^{-2}, -q^{2k}s^{-2}t^{-2}; q^2)_\infty}{(-1, -q^2, q^2s^{-2}, q^2s^{-2}; q^2)_\infty} \\ \sum_{n=-\infty}^{\infty} \left(\frac{uq^2}{s^2}\right)^n \frac{(q^{2+2n-2k}; q^2)_\infty}{(-q^{2n}; q^2)_\infty} {}_2\varphi_1 \left( \begin{matrix} -q^{2-2k}t^2, -q^{2n} \\ q^{2+2n-2k} \end{matrix}; q^2, -\frac{q^{2k}}{s^2t^2} \right) \\ \times \frac{(q^{2+2n-2k}; q^2)_\infty}{(-q^{2n}t^{-2}; q^2)_\infty} {}_2\varphi_1 \left( \begin{matrix} -q^{2n}t^{-2}, -q^{2-2k} \\ q^{2+2n-2k} \end{matrix}; q^2, -\frac{q^{2k}}{s^2} \right)$$

Recall the identity, see [24, Prop. 2.2, with  $n = 0$ ,  $w = tx^{-1}y^{-1}$ ],

$$\sum_{m=-\infty}^{\infty} w^m \frac{(q^{m+1}; q)_\infty}{(aq^m; q)_\infty} {}_2\varphi_1 \left( \begin{matrix} aq^m, b \\ q^{m+1} \end{matrix}; q, -x \right) \frac{(q^{m+1}; q)_\infty}{(cq^m; q)_\infty} {}_2\varphi_1 \left( \begin{matrix} cq^m, d \\ q^{m+1} \end{matrix}; q, -y \right) = \\ \frac{(q, q; q)_\infty}{(a, c; q)_\infty} \sum_{p=0}^{\infty} (-x)^p \frac{(a, b; q)_p}{(q, q; q)_p} {}_2\varphi_1 \left( \begin{matrix} q^{-p}, d \\ q^{1-p}a^{-1} \end{matrix}; q, -\frac{qy}{aw} \right) {}_2\varphi_1 \left( \begin{matrix} q^{-p}, c \\ q^{1-p}b^{-1} \end{matrix}; q, -\frac{qw}{bx} \right).$$

After specialising  $a = -y/w$ ,  $c = -x/w$  we can use the  $q$ -Chu-Vandermonde sums [17, (1.5.2), (1.5.3)] to sum the two terminating  ${}_2\varphi_1$ -series in the summand on the right hand side. This gives

$$\sum_{m=-\infty}^{\infty} \frac{w^m (q^{m+1}; q)_\infty}{(-yq^m/w; q)_\infty} {}_2\varphi_1 \left( \begin{matrix} -\frac{yq^m}{w}, b \\ q^{m+1} \end{matrix}; q, -x \right) \frac{(q^{m+1}; q)_\infty}{(-xq^m/w; q)_\infty} {}_2\varphi_1 \left( \begin{matrix} -\frac{xq^m}{w}, d \\ q^{m+1} \end{matrix}; q, -y \right) \\ = \frac{(q, q; q)_\infty}{(-y/w, -x/w; q)_\infty} {}_2\varphi_1 \left( \begin{matrix} -yd/w, -bx/w \\ q \end{matrix}; q, w \right),$$

valid for  $1 > |w| > |xy|$ . Indeed, using (2.10) we see that

$$\frac{(q^{m+1}; q)_\infty}{(-xq^m/w; q)_\infty} {}_2\varphi_1 \left( \begin{matrix} -\frac{xq^m}{w}, d \\ q^{m+1} \end{matrix}; q, -y \right) = \begin{cases} \mathcal{O}(1), & m \rightarrow \infty, \\ \mathcal{O}((-x)^{-m}), & m \rightarrow -\infty. \end{cases}$$

Use this identity in base  $q^2$  with  $b = -q^{2-2k}t^2$ ,  $d = -q^{2-2k}$ ,  $x = q^{2k}s^{-2}t^{-2}$ ,  $y = q^{2k}s^{-2}$ , and  $w = s^{-2}$ , so we specialise  $u = q^{-2}$ . Note that  $s^{-2} < 1$  since  $|s| > 1$ , and  $|xy| = q^{4k}s^{-4}t^{-2} < q^2s^{-2}$ , since we have to evaluate at the discrete mass point  $\mu(-q^{1-2k}st)$  so that  $|q^{1-2k}st| > 1$ . So we may use this identity to see that the sum in (B.8) for  $u = q^{-2}$  equals

$$s^{-2k} \frac{(q^2, q^2; q^2)_\infty}{(-q^{2k}, -q^{2k}t^{-2}; q^2)_\infty} {}_2\varphi_1 \left( \begin{matrix} q^2, q^2 \\ q^2 \end{matrix}; q^2, s^{-2} \right) = \frac{s^{-2k}}{1 - s^{-2}} \frac{(q^2, q^2; q^2)_\infty}{(-q^{2k}, -q^{2k}t^{-2}; q^2)_\infty}.$$

Using this gives

$$R_{q^{-2}}(\mu(-stq^{1-2k}); s, t|q^2) = \frac{(-t^{-2}, -q^2t^2, -q^{2k}s^{-2}, -q^{2k}s^{-2}t^{-2}, q^2, q^2; q^2)_\infty}{(-1, -q^2, q^2s^{-2}, q^2s^{-2}, -q^{2k}, -q^{2k}t^{-2}; q^2)_\infty} \frac{s^{-2k}}{1 - s^{-2}}.$$

Note that specialising  $e^{i\theta} = -q^{1-2k}st$  in Lemma 5.5 results in the same answer after manipulating theta-products, since the  ${}_8W_7$ -series is not singular for this value and the factor in front of the  ${}_8W_7$ -series is zero. Taking into account the first term gives the result.  $\square$

**B.3 Proof of Lemma 6.4.** The proof of Lemma 6.4 is similar to the proof given in the first subsection of this appendix, but it is simpler. It again uses the Askey-Wilson polynomials and a corresponding  $q$ -integral representation. However, we have to distinguish between the principal unitary series on the one hand and the complementary and the strange series on the other hand in some derivations.

We first observe that  $|\langle v_t^\bullet, e_n \rangle|$  behaves as  $|s/q|^n$  as  $n \rightarrow -\infty$  and as  $|s|^{-n}$  as  $n \rightarrow \infty$ . This follows from the results from Appendix A. It follows that the doubly infinite sum of Lemma 6.4 is absolutely convergent in the annulus  $|q/st| < |z| < |st/q|$ .

The analogue of Lemma B.1 is the following.

**Lemma B.4.** *With the notation of §6.2 and §6.3 and with the assumptions of Lemma 6.4 and assuming  $n \leq 0$  we have*

$$\langle v_t^\bullet, e_n \rangle \langle e_n, v_s^\bullet \rangle = (st)^n q^{-2n} \sum_{m=0}^{\infty} q^{-2nm} p_m(q^{1+2\lambda}; q, qs^{-2}, qt^{-2}, q|q^2)$$

for  $\bullet \in \{P, C, S\}$  using the notation (B.2) for the Askey-Wilson polynomials.

*Proof.* The proof is similar to the proof of Lemma B.1, but we have to choose the right form of the  ${}_2\varphi_1$ -series in order to have the  ${}_4\varphi_3$ -series balanced. We start with, cf. (6.4),

$$(B.9) \quad \langle v_t^\bullet, e_n \rangle \langle e_n, v_s^\bullet \rangle = (st)^n q^{-4n(1+\Re\lambda)} \sqrt{\frac{(q^{-2\lambda+2n}, q^{-2\bar{\lambda}+2n}; q^2)_\infty}{(q^{2\lambda+2n+2}, q^{2\bar{\lambda}+2n+2}; q^2)_\infty}} \\ \times {}_2\varphi_1 \left( \begin{matrix} q^{2+2\lambda}t^{-2}, q^{2+2\lambda} \\ q^2t^{-2} \end{matrix}; q^2, q^{-2n-2\lambda} \right) {}_2\varphi_1 \left( \begin{matrix} q^{2+2\bar{\lambda}}s^{-2}, q^{2+2\bar{\lambda}} \\ q^2s^{-2} \end{matrix}; q^2, q^{-2n-2\bar{\lambda}} \right).$$

Note that for  $n \leq 0$  both  ${}_2\varphi_1$ -series are absolutely convergent, since  $|q^{2+2\lambda}| \leq q$  for  $\lambda$  as in Lemma 6.4. As in the proof of Lemma B.1 we rewrite the product of the two  ${}_2\varphi_1$ -series as

$$\sum_{m=0}^{\infty} q^{-2m(n+\bar{\lambda})} \frac{(q^{2+2\bar{\lambda}}s^{-2}, q^{2+2\bar{\lambda}}; q^2)_m}{(q^2, q^2s^{-2}; q^2)_m} {}_4\varphi_3 \left( \begin{matrix} q^{2+2\lambda}t^{-2}, q^{2+2\lambda}, q^{-2m}, q^{-2m}s^2 \\ q^2t^{-2}, q^{-2m-2\bar{\lambda}}s^2, q^{-2m-2\bar{\lambda}} \end{matrix}; q^2, q^{-4\Re\lambda} \right)$$

and the  ${}_4\varphi_3$ -series is balanced if  $\Re\lambda = -\frac{1}{2}$ , i.e. for  $\lambda$  corresponding to the principal unitary series. We can apply Sears's transformation [17, (2.10.4)] with  $a$  and  $d$  specialised to  $q^{2+2\lambda}$  and  $q^2t^{-2}$  to rewrite this as, using  $\Re\lambda = -\frac{1}{2}$ ,

$$(B.10) \quad \sum_{m=0}^{\infty} q^{-2mn} {}_4\varphi_3 \left( \begin{matrix} q^{-2m}, q^{2+2m}t^{-2}s^{-2}, q^{2+2\lambda}, q^{-\lambda} \\ q^2t^{-2}, q^2s^{-2}, q^2 \end{matrix}; q^2, q^2 \right)$$

and the  ${}_4\varphi_3$ -series is the Askey-Wilson polynomial as in the lemma. Note that the square root of  $q$ -shifted factorials in (B.9) reduces to 1 for  $\Re\lambda = -\frac{1}{2}$ . This proves the lemma for the principal unitary series.

For the complementary series and the strange series we use Heine's transformation formula [17, (1.4.6)];

$${}_2\varphi_1 \left( \begin{matrix} q^{2+2\bar{\lambda}}s^{-2}, q^{2+2\bar{\lambda}} \\ q^2s^{-2} \end{matrix} ; q^2, q^{-2n-2\bar{\lambda}} \right) = \frac{(q^{2+2\bar{\lambda}-2n}; q^2)_\infty}{(q^{-2n-2\bar{\lambda}}; q^2)_\infty} {}_2\varphi_1 \left( \begin{matrix} q^{-2\bar{\lambda}}, q^{-2\bar{\lambda}}s^{-2} \\ q^2s^{-2} \end{matrix} ; q^2, q^{2+2\bar{\lambda}-2n} \right).$$

The theta product identity (2.9) can now be used to rewrite (B.9) as

$$(B.11) \quad \langle v_t^\bullet, e_n \rangle \langle e_n, v_s^\bullet \rangle = (ts)^n q^{-2n(1-\lambda+\bar{\lambda})} \sqrt{\frac{(q^{-2\lambda+2n}, q^{2\bar{\lambda}+2n+2}; q^2)_\infty}{(q^{2\lambda+2n+2}, q^{-2\bar{\lambda}+2n}; q^2)_\infty}} \\ \times {}_2\varphi_1 \left( \begin{matrix} q^{2+2\lambda}t^{-2}, q^{2+2\lambda} \\ q^2t^{-2} \end{matrix} ; q^2, q^{-2n-2\lambda} \right) {}_2\varphi_1 \left( \begin{matrix} q^{-2\bar{\lambda}}, q^{-2\bar{\lambda}}s^{-2} \\ q^2s^{-2} \end{matrix} ; q^2, q^{2+2\bar{\lambda}-2n} \right).$$

The product of the two  ${}_2\varphi_1$ -series can be written as

$$\sum_{m=0}^{\infty} q^{-2m(n+\lambda)} \frac{(q^{2+2\lambda}t^{-2}, q^{2+2\lambda}; q^2)_m}{(q^2, q^2t^{-2}; q^2)_m} {}_4\varphi_3 \left( \begin{matrix} q^{-2\bar{\lambda}}, q^{-2\bar{\lambda}}s^{-2}, q^{-2m}, q^{-2pt^2} \\ q^2s^{-2}, q^{-2m-2\lambda}t^2, q^{-2m-2\lambda} \end{matrix} ; q^2, q^{2+2\bar{\lambda}-2\lambda} \right)$$

and the  ${}_4\varphi_3$ -series is balanced if  $q^{2\lambda} = q^{2\bar{\lambda}}$ . Since this is the case for the complementary series and the strange series, we apply once more Sears's transformation [17, (2.10.4)] with  $a$  and  $d$  specialised to  $q^{-2\bar{\lambda}}$  and  $q^2s^{-2}$  to rewrite this sum as (B.10) for  $\lambda$  satisfying  $q^{2\lambda} = q^{2\bar{\lambda}}$ . Observe that  $q^{2\lambda} = q^{2\bar{\lambda}}$  makes the square root of  $q$ -shifted factorials in (B.11) equal to 1. This proves the result for the complementary series and the strange series.  $\square$

We next employ the following  $q$ -integral representation for the Askey-Wilson polynomials of (B.2);

$$(B.12) \quad p_m(x; a, b, c, d|q) = (A(x; a, b, c, d|q))^{-1} \frac{(bc; q)_m}{(ad; q)_m} \\ \times \int_{qx/d}^{q/xd} \frac{(dux, du/x, abcd u/q; q)_\infty}{(adu/q, bdu/q, cdu/q; q)_\infty} \frac{(q/u; q)_m}{(abcd u/q; q)_m} \left( \frac{adu}{q} \right)^m d_q u, \\ A(x; a, b, c, d|q) = \frac{q(1-q)}{d(x-x^{-1})} \frac{(x^2, x^{-2}, q, ab, ac, bc; q)_\infty}{(ax, a/x, bx, b/x, cx, c/x; q)_\infty},$$

where the  $q$ -integral is defined by, cf. (4.6),

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x, \quad \int_0^c f(x) d_q x = (1-q)c \sum_{n=0}^{\infty} f(cq^n) q^n,$$



cf. §4.2. The  $q$ -integral representation is in Exercise 7.34 of [17]. The proof consists of rewriting the  $q$ -integral into the form [17, (2.10.19)], which can be done in such a way that the very-well-poised  ${}_8\varphi_7$ -series is terminating. The terminating very-well-poised  ${}_8\varphi_7$ -series can then be rewritten as a terminating balanced  ${}_4\varphi_3$ -series by Watson's transformation formula [17, (2.5.1)], which can be recognised as an Askey-Wilson polynomial in the form (B.2).

Using (B.12) and Lemma B.4 we have for  $n \leq 0$

$$(B.13) \quad \langle v_t^\bullet, e_n \rangle \langle e_n, v_s^\bullet \rangle = (ts)^n q^{-2n} A \int_{q^{2+2\lambda}}^{q^{-2\lambda}} \frac{(uq^{2+2\lambda}, uq^{-2\lambda}, q^2 s^{-2} t^{-2} u; q^2)_\infty}{(s^{-2} u, ut^{-2}, u; q^2)_\infty} \\ \times {}_2\varphi_1 \left( \begin{matrix} q^2 s^{-2} t^{-2}, q^2/u \\ q^2 s^{-2} t^{-2} u \end{matrix}; q^2, uq^{-2n} \right) d_{q^2} u,$$

where  $A^{-1} = A(q^{1+2\lambda}; qs^{-2}, q, qt^{-2}; q|q^2)$ . Interchanging  $q$ -integrating and summation is justified, since all sums are absolutely convergent for  $n \leq 0$  because  $|q^{2+2\lambda}| \leq q$  and  $|q^{-2\lambda}| \leq q$ . We can next use (B.13) for the expression to extend the left hand side to the case  $n > 0$  by using the analytic continuation of the  ${}_2\varphi_1$ -series in the  $q$ -integral of (B.13). The analytic continuation of the  ${}_2\varphi_1$ -series is given by [17, (4.3.2)], a formula we already used for the  $c$ -function expansion of (A.10). In this particular case the second term vanishes for  $n \in \mathbb{Z}$  and the factor in front of the remaining  ${}_2\varphi_1$ -series can be simplified using the theta product identity (2.9). This gives

$$(B.14) \quad {}_2\varphi_1 \left( \begin{matrix} q^2 s^{-2} t^{-2}, q^2/u \\ q^2 s^{-2} t^{-2} u \end{matrix}; q^2, uq^{-2n} \right) = (q^2 s^{-2} t^{-2})^n {}_2\varphi_1 \left( \begin{matrix} q^2 s^{-2} t^{-2}, q^2/u \\ q^2 s^{-2} t^{-2} u \end{matrix}; q^2, uq^{2n} \right).$$

For another way to see this, rewrite the left hand side using Heine's transformation [17, (1.4.5)] to recognise the  $q$ -Bessel coefficient behaviour. Next (2.10) provides the requested relation.

**Lemma B.5.** *For  $z$  in the annulus  $|q/st| < |z| < |st/q|$  we have*

$$\sum_{n=-\infty}^0 (ts)^n q^{-2n} (qz)^n {}_2\varphi_1 \left( \begin{matrix} q^2 s^{-2} t^{-2}, q^2/u \\ q^2 s^{-2} t^{-2} u \end{matrix}; q^2, uq^{-2n} \right) + \\ \sum_{n=1}^{\infty} (ts)^{-n} (qz)^n {}_2\varphi_1 \left( \begin{matrix} q^2 s^{-2} t^{-2}, q^2/u \\ q^2 s^{-2} t^{-2} u \end{matrix}; q^2, uq^{2n} \right) \\ = \frac{(1 - q^2/s^2 t^2)}{(1 - q/zst)(1 - qz/st)} \frac{(q^4/s^2 t^2, q^2, qu/zst, quz/st; q^2)_\infty}{(q^3/zst, q^3 z/st, q^2 u/s^2 t^2, u; q^2)_\infty}.$$

*Proof.* We can write the left hand side as

$$\sum_{j=0}^{\infty} \frac{(q^2 s^{-2} t^{-2}, q^2/u; q^2)_j}{(q^2, q^2 s^{-2} t^{-2} u; q^2)_j} u^j \left( \sum_{n=-\infty}^0 (ts)^n q^{-2n-2nj} (qz)^n + \sum_{n=1}^{\infty} (ts)^{-n} (qz)^n q^{2nj} \right).$$

Both sums in parantheses are geometric sums and absolutely convergent for  $z$  in the annulus. The sums in parantheses equal

$$\frac{(1 - q^{2+4j}/s^2t^2)}{(1 - q^{2j+1}/zst)(1 - q^{2j+1}z/st)} = \frac{(1 - q^2/s^2t^2)}{(1 - q/zst)(1 - qz/st)} \frac{(1 - q^{2+4j}/s^2t^2) (q/zst, qz/st; q^2)_j}{(1 - q^2/s^2t^2) (q^3/zst, q^3z/st; q^2)_j}.$$

Plugging this back gives the left hand side as a very-well-poised  ${}_6\varphi_5$ -series;

$$\frac{(1 - q^2/s^2t^2)}{(1 - q/zst)(1 - qz/st)} {}_6W_5(q^2/s^2t^2; qz/st, q/zst, q^2/u; q^2, u) = \frac{(1 - q^2/s^2t^2)}{(1 - q/zst)(1 - qz/st)} \frac{(q^4/s^2t^2, q^2, qu/zst, quz/st; q^2)_\infty}{(q^3/zst, q^3z/st, q^2u/s^2t^2, u; q^2)_\infty},$$

where we used the summation formula [17, (2.7.1)].  $\square$

Combining (B.13) and Lemma B.5 we see that for  $z$  in the annulus as in Lemma B.5 we have

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \langle v_t^\bullet, e_n \rangle \langle e_n, v_s^\bullet \rangle q^n z^n &= A \frac{(1 - q^2/s^2t^2)}{(1 - q/zst)(1 - qz/st)} \frac{(q^4/s^2t^2, q^2; q^2)_\infty}{(q^3/zst, q^3z/st; q^2)_\infty} \\ &\quad \times \int_{q^{2+2\lambda}}^{q^{-2\lambda}} \frac{(uq^{2+2\lambda}, uq^{-2\lambda}, qu/zst, quz/st; q^2)_\infty}{(s^{-2}u, ut^{-2}, u, u; q^2)_\infty} d_{q^2}u. \end{aligned}$$

The  $q$ -integral is of the same type as used for the Askey-Wilson polynomial, and it can be explicitly evaluated in terms of a very-well poised  ${}_8\varphi_7$ -series by [17, (2.10.19)]. The  $q$ -integral is equal to

$$\begin{aligned} q^{-2\lambda}(1 - q^2) &\frac{(q^2, q^{-4\lambda}, q^{2+4\lambda}, q^2/t^2, q^2/s^2, q^2/s^2t^2, q^{1-2\lambda}/zst, q^{1-2\lambda}z/st; q^2)_\infty}{(q^{2+2\lambda}/s^2, q^{2+2\lambda}/t^2, q^{2+2\lambda}, q^{-2\lambda}/s^2, q^{-2\lambda}/t^2, q^{-2\lambda}, q^{-2\lambda}, q^{2-2\lambda}/s^2t^2; q^2)_\infty} \\ &\times {}_8W_7(q^{-2\lambda}/s^2t^2; q^{-2\lambda}/s^2, q^{-2\lambda}/t^2, q^{-2\lambda}, qz/st, q/zst; q^2, q^{2+2\lambda}). \end{aligned}$$

Plugging this back in and using the value for  $A$  proves Lemma 6.4 for  $z$  in the annulus  $|q/st| < |z| < |st/q|$ .  $\square$

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